AN

ELEMENTARY TREATISE

ON

PLANE TRIGONOMETRY,

WITH ITS APPLICATIONS TO

HEIGHTS AND DISTANCES, NAVIGATION,

AND

SURVEYING.

By BENJAMIN PEIRCE, A. M.,

UNIVERSITY PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY IN HARVARD UNIVERSITY.

CAMBRIDGE:

JAMES MUNROE AND COMPANY,

BOOKSELLERS TO THE UNIVERSITY.

BOSTON:

134 WASHINGTON STREET.

MDCCC XXXV.

[illegible]

O

AN ELEMENTARY TREATISE ON PLANE TRIGONOMETRY,

WITH ITS APPLICATIONS TO

HEIGHTS AND DISTANCES, NAVIGATION,

AND

SURVEYING.

By BENJAMIN PEIRCE, A. M.,

UNIVERSITY PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY IN HARVARD UNIVERSITY.

CAMBRIDGE & BOSTON:

JAMES MUNROE AND COMPANY.

MDCCCXXXV.

CAMBRIDGE PRESS:
METCALF, TORRY, AND BALLOU.

TABLE OF CONTENTS.

CHAPTER I.

CHAPTER II.

SECTION II.

CHAPTER III.

CHAPTER IV.

SECTION I.

SECTION II.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

SECTION I.

SECTION II.

PLANE TRIGONOMETRY.

CHAPTER I.

General Principles of Plane Trigonometry.

SECTION I.

Definition and Objects of Plane Trigonometry.

1. *Trigonometry* is the science which treats of angles and triangles. The solution of plane triangles is the principal object of the present elementary treatise, and hardly any theorems will be given in relation to angles which are not subsidiary to this purpose.

2. *To solve a Triangle* is to calculate certain of its sides and angles when the others are known. Now it has been proved in Geometry that, when three of the six parts of a triangle are given, the triangle can be constructed, provided one at least of the given parts is a side. In these cases, then, the unknown parts of the triangle can be determined geometrically, and it may readily be inferred that they can also be determined algebraically.

3. But a great difficulty is met with on the very threshold of the attempt to apply the calculus to triangles. It arises from the circumstance that two

kinds of quantities are to be introduced into the same formulas, sides, and angles. These quantities are not only of an entirely different species, but the law of their relative increase and decrease is so complicated, that they cannot be determined from each other by any of the common operations of Algebra.

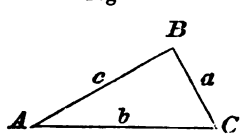
4. To diminish this difficulty as much as possible, every method has been taken to compare triangles with each other, and the solution of all triangles has been reduced to that of a *Limited Series of Right-angled Triangles.*

SECTION II.

Principles on which the Solution of all Triangles is reduced to that of a Limited Series of Right-angled Triangles.

5. It is a well known proposition of Geometry, that, in all triangles, which are equiangular with respect to each other, the ratios of the homologous sides are also equal. If, then, a series of dissimilar triangles were constructed containing every possible variety of angles; and, if the angles and the ratios of the sides were all known, we should find it easy to calculate every case of triangles. Suppose, for instance, that in the triangle ABC (fig. 1.), the sides of which we shall denote by the small letters a, b, c, respectively opposite to the angles A, B, C, there are given the two sides b and c and the included angle A, to find the side a and the angles B and C. We are to look through the series of calculated tri-

Fig. 1.



angles, till we find one which has an angle equal to A, and the ratio of the including sides equal to that of b and c. As this triangle is similar to ABC, its angles and the ratio of its sides must also be those of the triangle ABC, which is therefore completely determined. For, to find the side a, we have only to multiply the ratio which we have found of b to a, that is, the fraction $\frac{a}{b}$ by the side b or the ratio $\frac{a}{c}$ by the side c.

6. A series of calculated triangles is not, however, needed for any other than right-angled triangles. For every oblique triangle is either the sum or the difference of two right triangles; and the sides and angles of the oblique triangle are the same with those of the right triangles, or may be obtained from them by addition or by subtraction. Thus the triangle ABC is the sum (fig. 2.) or the difference (fig. 3.) of the two right triangles ABP and BPC. In both figures the sides AB, BC, and the angle A belong at once to the oblique and the right triangles, and so does the angle BCA (fig. 2.) or its supplement (fig. 3.); while the angle ABC is the sum (fig. 2.), or the difference (fig. 3.) of ABP and PBC; and the side AC is the sum (fig. 2.), or the difference (fig. 3.) of AP and PC.

Fig. 2.

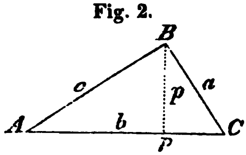
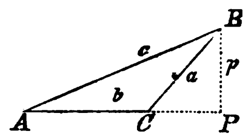


Fig. 3.



7. But, as even a series of right triangles, which should contain every variety of angle, would be un-

limited, it could never be constructed or calculated. Fortunately, such a series is not required; and it is sufficient for all practical purposes to calculate a series in which the successive angles differ only by a minute, or, at the least, by a second. The other triangles can be obtained, when needed, by that simple principle of interpolation made use of to obtain the intermediate logarithms from those given in the tables. We shall illustrate this principle more at length in the introduction to the use of the *Trigonometrical Tables.*

CHAPTER II.

On the Calculation of the Tables of Sines, Cosines, &c.

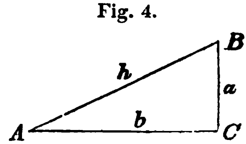
SECTION I.

Definitions. Formulas expressing Relations between the different Trigonometrical Functions of an Angle.

8. *Definitions.* Confining ourselves, for the present, to right triangles, we now proceed to introduce some terms, for the purpose of giving simplicity and brevity to our language.

The *Sine* of an angle is the quotient obtained by dividing the leg opposite it in a right triangle by the hypothenuse. Thus, if we denote (fig. 4.) the legs BC and AC by the letters a and b and the hypothenuse AB by the letter h, we have

Fig. 4.



(1) $$\sin. A = \frac{a}{h}, \quad \sin. B = \frac{b}{h}.$$

The *Tangent* of an angle is the quotient obtained by dividing the leg opposite it in a right triangle, by the adjacent leg. Thus

$$\text{tang. } A = \frac{a}{b}, \quad \text{tang. } B = \frac{b}{a}. \tag{2}$$

The *Secant* of an angle is the quotient obtained by dividing the hypothenuse by the leg adjacent to the angle. Thus

$$\text{sec. } A = \frac{h}{b}, \quad \text{sec. } B = \frac{h}{a}. \tag{3}$$

The *Cosine*, *Cotangent*, and *Cosecant* of an angle are respectively the sine, tangent, and secant of its complements. (4)

9. *Corollary.* Since the two acute angles of a right triangle are complements of each other, the sine, tangent, and secant of the one must be the cosine, cotangent, and cosecant of the other. Thus (fig. 4.)

$$\left.\begin{aligned}
\text{sin. } A &= \text{cos. } B = \frac{a}{h},\\
\text{cos. } A &= \text{sin. } B = \frac{b}{h},\\
\text{tang. } A &= \text{cotan. } B = \frac{a}{b},\\
\text{cotan. } A &= \text{tang. } B = \frac{b}{a},\\
\text{sec. } A &= \text{cosec. } B = \frac{h}{b},\\
\text{cosec. } A &= \text{sec. } B = \frac{h}{a}.
\end{aligned}\right\} \tag{5}$$

10. *Corollary.* By inspecting the preceding equations (5), we perceive that the sine and cosecant of

an angle are reciprocals of each other; as are also the cosine and secant, and also the tangent and cotangent. So that

$$(6)\quad \begin{cases} \text{cosec. } A \times \text{sin. } A = \dfrac{h}{a} \times \dfrac{a}{h} = \dfrac{ah}{ah} = 1, \\ \text{sec. } A \times \text{cos. } A = \dfrac{h}{b} \times \dfrac{b}{h} = \dfrac{bh}{bh} = 1, \\ \text{tang. } A \times \text{cotan. } A = \dfrac{a}{b} \times \dfrac{b}{a} = \dfrac{ab}{ab} = 1; \end{cases}$$

whence

$$(7)\quad \begin{cases} \text{cosec. } A = \dfrac{1}{\text{sin. } A}, \text{ or sin. } A = \dfrac{1}{\text{cosec.} A}; \\ \text{sec. } A = \dfrac{1}{\text{cos. } A}, \text{ or cos. } A = \dfrac{1}{\text{sec. } A}; \\ \text{cotan. } A = \dfrac{1}{\text{tang.} A}, \text{ or tang. } A = \dfrac{1}{\text{cotan.} A}. \end{cases}$$

As soon, then, as the sine, cosine, and tangent of an angle are known, their reciprocals the cosecant, secant, and cotangent may easily be obtained.

11. *Problem.* To find the tangent when the sine and cosine of an angle are known.

Solution. The quotient of sin. A divided by cos. A is by equation (5)

$$(8)\quad \frac{\text{sin. } A}{\text{cos. } A} = \frac{a}{h} \div \frac{b}{h} = \frac{ah}{bh} = \frac{a}{b}.$$

But by (5)

$$(9)\quad \text{tang. } A = \frac{a}{b};$$

hence, from (8) and (9)

$$(10)\quad \text{tang. } A = \frac{\text{sin. } A}{\text{cos. } A}.$$

12. *Corollary.* Since the cotangent is the reciprocal of the tangent, we have

$$\text{cotan. } A = \frac{\cos. A}{\sin. A}. \tag{11}$$

13. *Problem.* To find the cosine of an angle when its sine is known.

Solution. We have, by the Pythagorean proposition, in the right triangle ABC (fig. 4.)

$$a^2 + b^2 = h^2. \tag{12}$$

But by (5)

$$\left.\begin{aligned} &(\sin. A)^2 + (\cos. A)^2 = \frac{a^2}{h^2} + \frac{b^2}{h^2} = \frac{a^2+b^2}{h^2} = \frac{h^2}{h^2} = 1, \\ &\text{or} \\ &(\sin. A)^2 + (\cos. A)^2 = 1; \end{aligned}\right\} \tag{13}$$

that is, *the sum of the squares of the sine and cosine is equal to unity.* Hence

$$(\cos. A)^2 = 1 - (\sin. A)^2, \tag{14}$$

$$\cos. A = \sqrt{1 - (\sin. A)^2}. \tag{15}$$

14. *Corollary.* Since by (12)

$$h^2 - a^2 = b^2, \tag{16}$$

we have by (5)

$$\left.\begin{aligned} &(\sec. A)^2 - (\text{tang. } A)^2 = \frac{h^2}{b^2} - \frac{a^2}{b^2} = \frac{h^2-a^2}{b^2} = \frac{b^2}{b^2} = 1, \\ &\text{or} \\ &(\sec. A)^2 - (\text{tang. } A)^2 = 1; \end{aligned}\right\} \tag{17}$$

whence $(\sec. A)^2 = 1 + (\text{tang. } A)^2.$ $\quad (18)$

15. *Corollary.* Since by (12)

$$h^2 - b^2 = a^2 \tag{19}$$

we have by (5)

(20) $$\begin{cases}(\text{cosec.}\,A)^2-(\text{cotan.}\,A)^2=\dfrac{h^2}{a^2}-\dfrac{b^2}{a^2}=\dfrac{h^2-b^2}{a^2}=\dfrac{a^2}{a^2}=1,\\ \text{or}\\ (\text{cosec.}\,A)^2-(\text{cotan.}\,A)^2=1\,;\end{cases}$$

whence

(21) $$(\text{cosec.}\,A)^2=1+(\text{cotan.}\,A)^2.$$

16. *Scholium.* The whole difficulty of calculating the trigonometrical tables of sines and cosines, tangents and cotangents, secants and cosecants is, by the preceding proposition, reduced to that of calculating the sines alone. There are many methods of performing this process given in the Differential and Integral Calculus, but we shall now confine ourselves to a very simple though tedious one, as we merely wish to illustrate the possibility of the operation.

EXAMPLES.

1. Given the sine of the angle A, equal to 0.4568, to calculate its cosine, tangent, cotangent, secant, and cosecant.

By (15)

$$\cos.\,A=\sqrt{1-(\sin.A)^2}=\sqrt{(1+\sin.\,A)\,(1-\sin.\,A)}.$$

$(1+\sin.\,A=1.4568)$	0.16340
$(1-\sin.\,A=0.5432)$	9.73496
$(\cos.\,A)^2$	2|9.89836
$(\cos.\,A=0.8896)$	9.94918.

By (10) and (11)

$$\text{tang.}\,A=\frac{\sin.\,A}{\cos.\,A},\qquad \text{cotan.}\,A=\frac{\cos.\,A}{\sin.\,A}.$$

(sin. $A = 0.4568$)	9.65973 (ar. co.)	10.34027
(cos. $A = 0.8896$)(ar. co.)	10.05082	9.94918
(tang. $A = 0.5135$)	9.71055 (ar. co.)	10.28945
	cotan. $A = 1.9474$.	

By (7)

$$\text{sec. } A = \frac{1}{\text{cos. } A}, \qquad \text{cosec. } A = \frac{1}{\text{sin. } A}.$$

$$\text{log. sec. } A = -\text{log. cos. } A = 0.05082,$$
$$\text{sec. } A = 1.1241.$$
$$\text{log. cosec. } A = -\text{log. sin. } A = 0.34027,$$
$$\text{cosec. } A = 2.1891.$$

2. Given sin. $A = 0.1111$; find the cosine, tangent, cotangent, secant, and cosecant of A.

Ans. cos. $A = 0.9938$,
tang. $A = 0.1118$,
cotan. $A = 8.9452$,
sec. $A = 1.0062$,
cosec. $A = 9.0010$.

3. Given sin. $A = 0.9891$; find the cosine, tangent, cotangent, secant and cosecant of A.

Ans. cos. $A = 0.1472$,
tang. $A = 6.7173$,
cotan. $A = 0.1489$,
sec. $A = 6.7914$,
cosec. $A = 1.0110$.

SECTION II.

Calculation of the Table of Sines.

17. *Theorem.* The sine of a very small angle is nearly equal to the arc, which is its measure in a circle the radius of which is unity.

Fig. 5.

Demonstration. Take the very small angle C (fig. 5); from the vertex C as a centre, with a radius equal to unity, describe the arc AB. This arc may be considered as a straight line perpendicular to CA, and its approximation to a straight line is more nearly accurate the smaller the angle C is taken. In the right triangle CAB we have, then,

(22) $$\sin. C = \frac{AB}{CB} = \frac{AB}{\text{unity}} = AB,$$

which we wished to prove.

18. *Corollary.* We have also

(23) $$\cos. C = \frac{AC}{CB} = \frac{\text{unity}}{\text{unity}} = 1,$$

(24) $$\text{tang. } C = \frac{AB}{AC} = \frac{AB}{\text{unity}} = AB = \sin. C;$$

that is, *the cosine of a very small angle is equal to unity, and its tangent is equal to its sine.*

19. *Problem.* To find the sine of a very small angle.

Solution. Let the angle C (fig. 5) be the given angle, and suppose it to be exactly one minute. The arc AB must in this case be $\frac{1}{10800}$ of the semicircumference, of which unity or CA is radius. But the value of the semicircumference, of which unity is radius, has been found in Geometry to be 3.1415926. Therefore, by (22)

(25) $$\sin 1' = AB = \frac{3.1415926}{10800} = 0.00029.$$

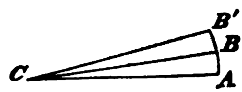
In the same way we might find the sine of any other

small angle, or we might, in preference, find it by the following proposition.

20. *Theorem.* The sines of very small angles are directly proportional to the angles themselves.

Demonstration. Let there be the two small angles, BCA and $B'CA$ (fig. 6). Draw the arc ABB' with the centre C, and the radius unity. Then, as angles are proportional to the arcs which measure them,

Fig. 6.



$$BCA : B'CA : : BA : B'A. \tag{26}$$

But by (22)

$$\sin.\ BCA = BA,\ \sin.\ B'CA = B'A; \tag{27}$$

which, substituted in (26), give

$$BCA : B'CA : : \sin.\ BCA : \sin.\ B'CA. \tag{28}$$

21. *Scholium.* The preceding proposition is limited to angles so small, that their arcs may be considered as straight lines. It is found in practice, that the angles may be as large as two degrees, provided the approximations are not carried beyond five places of decimals. The investigation of the sines of larger angles requires the introduction of some new formulas.

EXAMPLES.

1. Find the sine of 12′ 13″, knowing that

$$\sin.\ 1' = 0.00029.$$

Solution. By (28)

$$1' : 12'\ 13'' : : \sin.\ 1' : \sin.\ 12'\ 13'',$$

or

$$60'' : 733'' : : 0.00029 : \sin.\ 12'\ 13''.$$

Hence

$$\sin.\ 12'\ 13'' = \frac{733 \times 0.00029}{60} = 0.00354. \quad Ans.$$

2. Find the sine of 7′ 15″ knowing that

$$\sin.\ 1' = 0.00029.$$

Ans. $\sin.\ 7'\ 15'' = 0.00210.$

3. Find the sine of 2′ 31″ knowing that

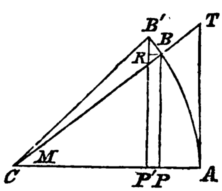
$$\sin.\ 1' = 0.00029.$$

Ans. $\sin.\ 2'\ 31'' = 0.00073.$

22. *Problem.* Given the sine of any angle, to find the sine of another angle which exceeds it by a very small quantity.

Solution. Let the given angle be BCA (fig. 7), which we will denote by the letter M; and let the angle whose sine is required be $B'CA$, exceeding the former by the small angle $B'CB$, which we will denote by the letter m; so that

Fig. 7.



(29) $$M = BCA, \qquad m = B'CB,$$
$$M + m = B'CA.$$

From the vertex C as a centre, with the radius unity, describe the arc ABB'. From the points B and B' let fall BP and $B'P'$ perpendicular to AC. In the right triangle BCP, we have by (1) and (29)

(30) $$\sin.\ M = \frac{BP}{BC} = BP,$$

(30′) or *the sine of an angle is equal to the perpendicular*

let fall from one extremity of the arc which measures it in the circle, whose radius is unity, upon the radius passing through the other extremity.

In the same way

$$\sin. B'CA = \sin. (M+m) = B'P'. \tag{31}$$

Moreover from (1) and (29)

$$\cos. M = \frac{PC}{BC} = PC; \tag{32}$$

or in the circle the radius of which is unity, *the cosine of an angle is equal to the part of the radius, perpendicular to the sine, included between the sine and the centre.* Hence (32′)

$$\cos. (M+m) = P'C. \tag{33}$$

Draw BR perpendicular to $B'P'$, and

$$\left.\begin{array}{l} B'P' = BP + B'R, \\ \text{or} \\ \sin. (M+m) = \sin. M + B'R. \end{array}\right\} \tag{34}$$

The triangles BCP and $BB'R$, having their sides perpendicular each to each, are similar and give the proportion

$$BC : BB' :: CP : B'R. \tag{35}$$

But, by (22),

$$BB' = \sin. m. \tag{36}$$

Hence

$$BC = 1 : \sin. m :: \cos. M : B'R; \tag{37}$$

$$\text{and } B'R = \sin. m. \cos. M, \tag{38}$$

which, being substituted in (34), gives the formula

$$\sin. (M+m) = \sin. M + \sin. m. \cos. M. \tag{39}$$

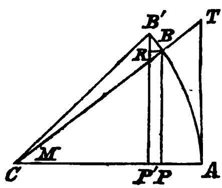
23. *Corollary.* If m were $1'$, (39) would become

(40) $$\sin.\,(M+1') = \sin.\,M + \sin.\,1'.\,\cos.\,M,$$
$$= \sin.\,M + 0.00029\,\cos.\,M.$$

We may, by this formula, find the sine of 2′ from that of 1′, thence that of 3′, then of 4′, of 5′, &c.; to the sine of an angle of any number of degrees and minutes.

24. *Corollary.* We can, in a similar way, deduce the value of cos. $(M+m)$.

Fig. 7.



For, by (31),

(41) $$\cos.(M+m) = P'C = PC - PP',$$
$$= \cos.\,M - BR.$$

But the similar triangles

$BB'R$ and BCP give the proportion

(42) $$BC : BB' :: BP : BR,$$

or

(43) $$1 : \sin.\,m :: \sin.\,M : BR.$$

Hence

(44) $$BR = \sin.\,m.\,\sin.\,M,$$

and (41) becomes

(45) $$\cos.\,(M+m) = \cos.\,M - \sin.\,m.\,\sin.\,M,$$

and, if we make $m = 1'$, this equation becomes

$$\cos.\,(M+1') = \cos.\,M - \sin.\,1'.\,\sin.\,M,$$
(46) $$= \cos.\,M - 0.00029\,\sin.\,M.$$

25. *Corollary.* If, at the extremity of the radius AC, we erect the perpendicular AT, then by (2), (3), and (29),

(47) $$\text{tang.}\,M = \frac{AT}{CA} = AT;$$

$$\text{sec. } M = \frac{C\,T}{C\,A} = CT; \tag{48}$$

or, in a circle the radius of which is unity, *the secant of an angle is equal to the length of the radius, drawn through one extremity of the arc which measures the angle, and produced till it meets the tangent drawn through the other extremity.*

The trigonometrical tangent of an angle is equal to that part of the tangent, drawn through one extremity of the above arc, which is intercepted by the two radii which terminate the arc.

EXAMPLES.

1. Given the sine of 23° 28′ equal to 0.39822, to find the sine of 23° 29′.

Solution. We find the cosine of 23° 28′ by (15) to be

$$\text{cos. } 23° \, 28' = 0.91729.$$

Hence, by (40), making $M = 23° \, 28'$

$$\begin{aligned} \text{sin. } 23° \, 29' &= \text{sin. } 23° \, 28' + 0.00029 \text{ cos. } 23° \, 28', \\ &= 0.39822 + 0.00026, \\ &= 0.39848. \end{aligned}$$

Ans. sin. 23° 29′ = 0.39848.

2. Given the sine and cosine of 46° 58′ as follows sin. 46° 58′ = 0.73096, cos. 46° 58′ = 0.68042, find the sine of 46° 59′.

Ans. sin. 46° 59′ = 0.73116.

3. Given the sine and cosine of 11° 10′ as follows sin. 11° 10′ = 0.19366, cos. 11° 10′ = 0.98107, find the cosine of 11° 11′.

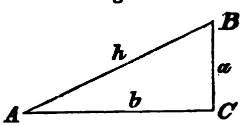
Ans. cos. 11° 11′ = 0.98101.

CHAPTER III.

On the Solution of Right Triangles.

26. *Problem.* To solve a right triangle, when the hypothenuse and one of the angles are known.

Fig. 4.



Solution. Given (fig. 4) the hypothenuse h and the angle A, to solve the triangle.

First. To find the other acute angle B, subtract the given angle from $90°$.

Secondly. To find the opposite side a, we have by (5)

(49) $$\sin. A = \frac{a}{h},$$

which, multiplied by h, gives

(50) $$a = h \sin. A;$$

or, by logarithms,

(51) $$\log. a = \log. h + \log. \sin. A.$$

Thirdly. To find the side b, we have by (5)

(52) $$\cos. A = \frac{b}{h},$$

which, multiplied by h, gives

(53) $$b = h \cos. A;$$

or, by logarithms,

(54) $$\log. b = \log. h + \log. \cos. A.$$

27. *Problem.* To solve a right triangle, when a leg and the opposite angle are known.

Solution. Given (fig. 4.) the leg a, and the opposite angle A, to solve the triangle.

First. The angle B is the complement of A.

Secondly. To find the hypothenuse h, we have by (50)

$$a = h \sin. A, \tag{55}$$

which, divided by sin. A, gives by (7)

$$h = \frac{a}{\sin. A} = a \operatorname{cosec.} A; \tag{56}$$

or, by logarithms,

$$\begin{aligned} \log. h &= \log. a + (\text{ar. co.}) \log. \sin. A \\ &= \log. a + \log. \operatorname{cosec.} A. \end{aligned} \tag{57}$$

Thirdly. To find the other leg b, we have by (5)

$$\operatorname{cotan.} A = \frac{b}{a}, \tag{58}$$

which, multiplied by a, gives

$$b = a \operatorname{cotan.} A; \tag{59}$$

or, by logarithms,

$$\log. b = \log. a + \log. \operatorname{cotan.} A. \tag{60}$$

28. *Problem.* To solve a right triangle, when a leg and the adjacent angle are known.

Solution. Given (fig. 4.) the leg a and the angle B, to solve the triangle.

First. The angle A is the complement of B.

Secondly. The other parts may be found by (56) and (59); or from the following equations, which are readily deduced from equations (5) and (7).

$$h = \frac{a}{\cos. B} = a \sec. B, \tag{61}$$

$$b = a \operatorname{tang.} B; \tag{62}$$

or, by logarithms,

(63) $$\log. h = \log. a + \log. \sec. B,$$

(64) $$\log. b = \log. a + \log. \text{tang}. B.$$

29. ***Problem.*** To solve a right triangle, when the hypothenuse and a leg are known.

Solution. Given (fig. 4.) the hypothenuse h and the leg a, to solve the triangle.

First. The angles A and B are obtained from equation (5)

(65) $$\sin. A = \cos. B = \frac{a}{h};$$

or, by logarithms,

(66) $$\log. \sin. A = \log. \cos. B = \log. a + (\text{ar. co.}) \log. h.$$

Secondly. The leg b is deduced from the Pythagorean property of the right triangle, which gives

(67) $$a^2 + b^2 = h^2,$$

whence

(68) $$b^2 = h^2 - a^2 = (h + a)(h - a),$$

(69) $$b = \sqrt{h^2 - a^2} = \sqrt{(h + a)(h - a)};$$

by logarithms,

(70) $$\log. b. = \tfrac{1}{2} \log. (h^2 - a^2) = \tfrac{1}{2} [\log. (h + a) + \log. (h - a)].$$

30. ***Problem.*** To solve a right triangle, when the two legs are known.

Solution. Given (fig. 4.) the legs a and b, to solve the triangle.

First. The angles are obtained from (5)

(71) $$\text{tang}. A = \text{cotan}. B = \frac{a}{b};$$

or, by logarithms,

$$\log.\ \text{tang}.\ A = \log.\ \text{cotan}.\ B = \log.\ a + (\text{ar. co.}) \log.\ b. \qquad (72)$$

Secondly. To find the hypothenuse, we have by (67)

$$h = \sqrt{a^2 + b^2}. \qquad (73)$$

Thirdly. An easier way of finding the hypothenuse is to make use of (56) or (61)

$$h = a \text{ cosec. } A = a \text{ sec. } B; \qquad (74)$$

or, by logarithms,

$$\log.\ h = \log.\ a. + \log.\ \text{cosec}.\ A = \log.\ a + \log.\ \text{sec}.\ B. \qquad (75)$$

EXAMPLES.

1. Given the hypothenuse of a right triangle equal to 49.58, and one of the acute angles equal to 54° 44′; to solve the triangle.

Solution. The other angle = 90° — 54° 44′ = 35° 16′. Then making $h = 49.58$, and $A = 54° 44'$; we have, by (50), by (53),

h 49.58	1.69531		1.69531
A 54° 44′	sin. 9.91194		cos. 9.76146
a 40.481	1.60725;	b 28.637	1.45677.

Ans. The other angle = 35° 16′;

The legs $= \begin{cases} 40.481, \\ 28.637. \end{cases}$

2. Given the hypothenuse of a right triangle equal to 54.571, and one of the legs equal to 23.479; to solve the triangle.

Solution. Making $h = 54.571$, $a = 23.479$; we have, by (65),

a 23.479		1.37068
h 54.571	(ar. co.)	8.26304
A 25° 29′ sin.	}	9.63372.
B 64° 31′ cos.		

By (70),

$h + a$	78.050	1.89237
$h - a$	31.092	1.49265
b^2		2 | 3.38502
b	49.262	1.69251.

Ans. The other leg $= 49.262$;

The angles $= \begin{cases} 25° \; 29', \\ 64° \; 31'. \end{cases}$

3. Given the two legs of a right triangle equal to 44.375, and 22.165; to solve the triangle.

Solution. Making $a = 44.375$, $b = 22.165$; we have, by (71), by (74),

a 44.375		1.64714		1.64714
b 22.165	(ar. co.)	8.65433		
A 63° 27′ tang. / B 26° 33′ cotan.	}	10.30147;	cosec. / sec. }	10.04837
			h 49.603	1.69551.

Ans. The hypothenuse $= 49.603$,

The angles $= \begin{cases} 63° \; 27', \\ 26° \; 33'. \end{cases}$

4. Given the hypothenuse of a right triangle equal to 37.364, and one of the acute angles equal to 12° 30′; to solve the triangle.

Ans. The other angle = 77° 30′;

The legs = $\begin{cases} 8.087, \\ 36.478. \end{cases}$

5. Given one of the legs of a right triangle equal to 14.548, and the opposite angle equal to 54° 24′; to solve the triangle.

Ans. The hypothenuse = 17.892;
The other leg = 10.415;
The other angle = 35° 36′.

6. Given one of the legs of a right triangle equal to 11.111, and the adjacent angle equal to 11° 11′, to solve the triangle.

Ans. The hypothenuse = 11.326;
The other leg = 2.197;
The other angle = 78° 49′.

7. Given the hypothenuse of a right triangle equal to 100, and one of the legs equal to 1, to solve the triangle.

Ans. The other leg = 99.995;

The angles = $\begin{cases} 0° \; 34', \\ 89° \; 26'. \end{cases}$

8. Given the two legs of a right triangle equal to 8.148, and 10.864, to solve the triangle.

Ans. The hypothenuse = 13.58;

The angles = $\begin{cases} 36° \; 52', \\ 53° \; 8'. \end{cases}$

CHAPTER IV.

Investigation of Elementary Trigonometrical Formulas.

SECTION I.

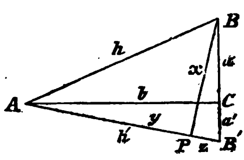
General Formulas.

31. The solution of oblique triangles requires the introduction of several trigonometrical formulas, which it is convenient to bring together and investigate all at once. The learner must not therefore be discouraged from reading the present chapter by not immediately understanding its end and use.

32. *Problem.* To find the sine of the sum of two angles.

Solution. Let the two angles be BAC and $B'AC$ (fig. 8), represented by the letters M and N. At any point C, in the line AC, erect the perpendicular BB'. From B let fall on AB' the perpendicular BP. Then represent the several lines, as follows,

Fig. 8.



(76) $$\begin{cases} a = BC,\ a' = B'C,\ b = AC, \\ h = AB,\ h' = AB',\ x = BP, \\ M = BAC, \quad N = B'AC. \end{cases}$$

Then, by (5),

$$\left.\begin{array}{ll} \sin. BAC = \sin. M = \frac{a}{h}, & \sin. N = \frac{a'}{h'}; \\ \cos. M = \frac{b}{h}, & \cos. N = \frac{b}{h'}. \end{array}\right\} \quad (77)$$

$$\sin. BAP = \sin. (M + N) = \frac{B\,P}{A\,B} = \frac{x}{h}. \quad (78)$$

Now the triangles BPB' and $B'AC$, being right-angled, and having the angle B' common, are equiangular and similar. Whence we derive the proportion

$$\left.\begin{array}{l} AB' : BB' :: AC : BP, \\ \text{or} \\ h' : a + a' :: b : x; \end{array}\right\} \quad (79)$$

whence

$$x = \frac{a\,b + a'\,b}{h'}, \quad (80)$$

and

$$\sin. (M + N) = \frac{x}{h} = \frac{a\,b + a'\,b}{h\,h'}. \quad (81)$$

The second member of this equation may be separated into factors, as follows,

$$\sin. (M + N) = \frac{a\,b}{hh'} + \frac{ba'}{hh'} \quad (82)$$

$$= \frac{a}{h} \cdot \frac{b}{h'} + \frac{b}{h} \cdot \frac{a'}{h'}. \quad (83)$$

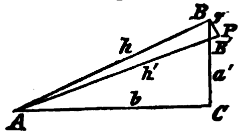
Substituting equations (77), we obtain

$$\sin. (M + N) = \sin. M \cos. N + \cos. M \sin. N. \quad (84)$$

33. *Problem.* To find the sine of the difference of two angles.

Solution. Let the two angles be BAC and $B'AC$ (fig. 9), represented by M and N. At any point C in the line AC erect the perpendicular $BB'C$.

Fig. 9.



From B let fall on AB' the perpendicular BP. Then, using the notation of (76), we have

(85) $$\sin. BAP = \sin. (M - N) = \frac{BP}{AB} = \frac{x}{h}.$$

The triangles $B'AC$ and $BB'P$ are similar, because they are right-angled, and the angles at B' are vertical and equal. Whence

(86) $$\left\{\begin{array}{l} AB' : BB' :: AC : BP, \\ \text{or} \\ h' : a - a' :: b : x; \end{array}\right.$$

whence

(87) $$x = \frac{ab - a'b}{h'},$$

and, by (85),

(88) $$\sin. (M - N) = \frac{x}{h} = \frac{ab - ba'}{hh'},$$

(89) $$= \frac{ab}{hh'} - \frac{ba'}{hh'},$$

(90) $$= \frac{a}{h} \cdot \frac{b}{h'} - \frac{b}{h} \cdot \frac{a'}{h'};$$

and from (77)

(91) $$\sin. (M - N) = \sin. M \cos. N - \cos. M \sin. N.$$

34. ***Problem.*** To find the cosine of the sum of two angles.

Solution. Making use of (fig. 8), with the notation of (76) and also the following

(92) $$y = AP,\ z = PB';$$

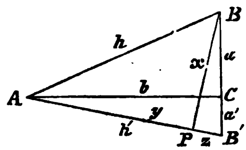
we have

(93) $$\cos. (M + N) = \frac{AP}{AB} = \frac{y}{h}.$$

But

(94) $$y = AB' - PB' = h' - z.$$

Fig. 8.



The similar triangles BPB' and $B'AC$, give the proportion

$$\left.\begin{array}{l} AB' : BB' : : B'C : B'P, \\ h' : a + a' : : a' : z ; \end{array}\right\} \quad (95)$$

or

whence

$$z = \frac{a\,a' + a'^{\,2}}{h'}, \quad (96)$$

and

$$y = h' - z = h' - \frac{a\,a' + a'^2}{h'}, \quad (97)$$

$$= \frac{h'^2 - a'^2 - a\,a'}{h'}. \quad (98)$$

But, from the right triangle $AB'C$,

$$h'^2 - a'^2 = (AB')^{\,2} - (B'C)^2 = (AC)^2 = b^2 ; \quad (99)$$

whence

$$y = \frac{b^2 - a\,a'}{h'} ; \quad (100)$$

and by (93),

$$\cos. (M + N = \frac{y}{h} = \frac{b^2 - a\,a'}{h\,h'}, \quad (101)$$

$$= \frac{b^2}{hh'} - \frac{aa'}{hh'}, \quad (102)$$

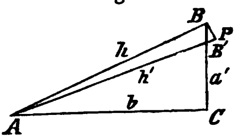
$$= \frac{b}{h} \cdot \frac{b}{h'} - \frac{a}{h} \cdot \frac{a'}{h'}. \quad (103)$$

Substituting equations (77),

$$\cos. (M + N) = \cos. M. \cos. N - \sin. M. \sin. N. \quad (104)$$

35. *Problem.* To find the cosine of the difference of two angles.

Fig. 9.



Solution. Making use of (fig. 9.) with the notation of (76) and (92), we have

(105) $$\cos. BAB' = \cos. (M - N) = \frac{A\,P}{A\,B} = \frac{y}{h}.$$

But

(106) $$y = AB' + PB' = h' + z.$$

The similar triangles $BB'P$ and $B'AC$ give the proportion

(107) $$\left\{ \begin{array}{l} AB' : BB' : : B'C : B'P, \\ \text{or} \\ h' : a - a' : : a' : z; \end{array} \right.$$

whence

(108) $$z = \frac{a\,a' - a'^2}{h'},$$

and

(109) $$y = h' + z = h' + \frac{a\,a' - a'^2}{h'},$$

(110) $$= \frac{h'^2 - a'^2 + a\,a'}{h'}.$$

But by (99),

(111) $$h'^2 - a'^2 = b^2.$$

Hence

(112) $$y = \frac{b^2 + a\,a'}{h'},$$

and by (105),

(113) $$\cos. (M - N) = \frac{y}{h} = \frac{b^2 + a\,a'}{h\,h'},$$

whence

$$\text{tang.}\,\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\cdot\text{tang.}\tfrac{1}{2}(A+B)=\frac{a-b}{a+b}\cdot\text{cot.}\tfrac{1}{2}C, \tag{234}$$

or, by logarithms,

$$\log.\ \text{tang.}\,\tfrac{1}{2}(A-B)=\log.\,(a-b)+(\text{ar. co.})\ \log.\,(a+b)+\log.\ \text{tang.}\,\tfrac{1}{2}(A+B). \tag{235}$$

The greater angle, which must be opposite the greater side, is then found by adding their half sum to their half difference; and the smaller angle by subtracting the half difference from the half sum. (236)

Secondly. The third side is found by (203), as follows;

$$\sin.\,A:\sin.\,C::a:c; \tag{237}$$

whence

$$c=\frac{a\sin.\,C}{\sin.\,A}, \tag{238}$$

or, by logarithms,

$$\log.\,c=\log.\,a+\log.\,\sin.\,C+(\text{ar. co.})\ \log.\,\sin.\,A. \tag{239}$$

EXAMPLES.

1. Given two sides of a triangle equal to 99.341 and 1.234, and their included angle equal to 169° 58′; to solve the triangle.

Solution. Making $a=99.341$, $b=1.234$; and

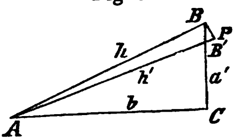
$$C=169^\circ\,58',\ \tfrac{1}{2}C=84^\circ\,59';$$

we have, by (235),

$a+b$	$=100.575$	(ar. co.)	7.99751
$a-b$	$=98.107$		1.99170
$\frac{1}{2}(A+B)$	$=5^\circ\ 1'$	tang.	8.94340
$\frac{1}{2}(A-B)$	$=4^\circ\,55'$	tang.	8.93261.
A	$=9^\circ\,56'$		
B	$=0^\circ\ 6'$		

35. *Problem.* To find the cosine of the difference of two angles.

Fig. 9.



Solution. Making use of (fig. 9.) with the notation of (76) and (92), we have

(105) $$\cos. BAB' = \cos. (M - N) = \frac{AP}{AB} = \frac{y}{h}.$$

But

(106) $$y = AB' + PB' = h' + z.$$

The similar triangles $BB'P$ and $B'AC$ give the proportion

(107) $$\left\{ \begin{array}{l} AB' : BB' :: B'C : B'P, \\ \text{or} \\ h' : a - a' :: a' : z; \end{array} \right.$$

whence

(108) $$z = \frac{a a' - a'^2}{h'}$$

and

(109) $$y = h' + z = h' + \frac{a a' - a'^2}{h'},$$

(110) $$= \frac{h'^2 - a'^2 + a a'}{h'}.$$

But by (99),

(111) $$h'^2 - a'^2 = b^2.$$

Hence

(112) $$y = \frac{b^2 + a a'}{h'},$$

and by (105),

(113) $$\cos. (M - N) = \frac{y}{h} = \frac{b^2 + a a'}{h h'},$$

whence

$$\text{tang.}\,\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\cdot\text{tang.}\tfrac{1}{2}(A+B)=\frac{a-b}{a+b}\cdot\text{cot.}\tfrac{1}{2}C, \tag{234}$$

or, by logarithms,

$$\text{log. tang.}\,\tfrac{1}{2}(A-B)=\text{log.}\,(a-b)+(\text{ar. co.})\ \text{log.}\,(a+b)+\text{log. tang.}\,\tfrac{1}{2}(A+B). \tag{235}$$

The greater angle, which must be opposite the greater side, is then found by adding their half sum to their half difference; and the smaller angle by subtracting the half difference from the half sum. (236)

Secondly. The third side is found by (203), as follows;

$$\sin.\,A:\sin.\,C::a:c; \tag{237}$$

whence

$$c=\frac{a\sin.\,C}{\sin.\,A}, \tag{238}$$

or, by logarithms,

$$\text{log.}\,c=\text{log.}\,a+\text{log. sin.}\,C+(\text{ar. co.})\ \text{log. sin.}\,A. \tag{239}$$

EXAMPLES.

1. Given two sides of a triangle equal to 99.341 and 1.234, and their included angle equal to 169° 58′; to solve the triangle.

Solution. Making $a=99.341$, $b=1.234$; and

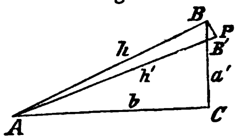
$$C=169^\circ\ 58',\ \tfrac{1}{2}C=84^\circ\ 59';$$

we have, by (235),

$a+b$	$=100.575$	(ar. co.)	7.99751
$a-b$	$=98.107$		1.99170
$\frac{1}{2}(A+B)$	$=5^\circ\ 1'$	tang.	8.94340
$\frac{1}{2}(A-B)$	$=4^\circ\ 55'$	tang.	8.93261.
A	$=9^\circ\ 56'$		
B	$=0^\circ\ 6'$		

35. *Problem.* To find the cosine of the difference of two angles.

Fig. 9.



Solution. Making use of (fig. 9.) with the notation of (76) and (92), we have

(105) $$\cos. BAB' = \cos. (M - N) = \frac{AP}{AB} = \frac{y}{h}.$$

But

(106) $$y = AB' + PB' = h' + z.$$

The similar triangles $BB'P$ and $B'AC$ give the proportion

(107) $$\left\{ \begin{array}{l} AB' : BB' : : B'C : B'P, \\ \text{or} \\ h' : a - a' : : a' : z; \end{array} \right.$$

whence

(108) $$z = \frac{a\,a' - a'^2}{h'}$$

and

(109) $$y = h' + z = h' + \frac{a\,a' - a'^2}{h'},$$

(110) $$= \frac{h'^2 - a'^2 + a\,a'}{h'}.$$

But by (99),

(111) $$h'^2 - a'^2 = b^2.$$

Hence

(112) $$y = \frac{b^2 + a\,a'}{h'},$$

and by (105),

(113) $$\cos. (M - N) = \frac{y}{h} = \frac{b^2 + a\,a'}{h\,h'},$$

whence

$$\text{tang.}\,\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\cdot\text{tang.}\tfrac{1}{2}(A+B)=\frac{a-b}{a+b}\cdot\text{cot.}\tfrac{1}{2}C, \quad (234)$$

or, by logarithms,

$$\text{log. tang.}\,\tfrac{1}{2}(A-B)=\text{log.}\,(a-b)+\text{(ar. co.)}\ \text{log.}\,(a+b)+\text{log. tang.}\,\tfrac{1}{2}(A+B). \quad (235)$$

The greater angle, which must be opposite the greater side, is then found by adding their half sum to their half difference; and the smaller angle by subtracting the half difference from the half sum. (236)

Secondly. The third side is found by (203), as follows;

$$\sin. A : \sin. C :: a : c; \quad (237)$$

whence

$$c=\frac{a \sin. C}{\sin. A}, \quad (238)$$

or, by logarithms,

$$\text{log.}\,c=\text{log.}\,a+\text{log. sin.}\,C+\text{(ar. co.) log. sin.}\,A. \quad (239)$$

EXAMPLES.

1. Given two sides of a triangle equal to 99.341 and 1.234, and their included angle equal to 169° 58′; to solve the triangle.

Solution. Making $a = 99.341$, $b = 1.234$; and

$$C = 169° 58', \ \tfrac{1}{2} C = 84° 59';$$

we have, by (235),

$a+b$	$= 100.575$	(ar. co.)	7.99751
$a-b$	$= 98.107$		1.99170
$\tfrac{1}{2}(A+B)$	$= 5° \ 1'$	tang.	8.94340
$\tfrac{1}{2}(A-B)$	$= 4° 55'$	tang.	8.93261.
A	$= 9° 56'$		
B	$= 0° \ 6'$		

$a = 99.341$			1.99713
$C = 169^\circ\ 58'$	sin.		9.24110
$A = 9^\circ\ 56'$	sin.	(ar. co.)	10.76321
$c = 100.433$			$2.00144.$

Ans. The third side $= 100.433$;

The other angles $= \begin{cases} 9^\circ\ 56', \\ 0^\circ\ \ 6'. \end{cases}$

2. Given two sides of a triangle equal to 0.121 and 5.421, and the included angle equal to $1^\circ\ 2'$; to solve the triangle.

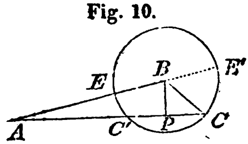
Ans. The other side $= 5.336$;

The other angles $= \begin{cases} 178^\circ\ 57', \\ 0^\circ\ \ 1'. \end{cases}$

59. *Theorem.* One side of a triangle is to the sum of the other two, as their difference is to the difference of the segments of the first side made by a perpendicular from the opposite vertex, if the per-
240) pendicular fall within the triangle; or to the difference of the distances of the extremity of the base from the foot of the perpendicular if it fall without the triangle

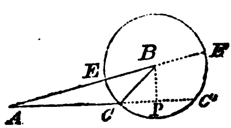
Demonstration. Let AC (figs. 10 and 11.) be the side of triangle ABC on which the perpendicular is let fall, and BP the perpendicular.

Fig. 10.



From B as a centre with a radius equal to BC the shorter of the other two sides, describe the circumference $CC'E'E$. Produce AB to E' and AC to C', if necessary.

Fig. 11.



Then, since AC and AB are secants, we have,

$$AC : AE' : : AE : AC'. \tag{241}$$

But

$$AE' = AB + BE' = AB + BC, \tag{242}$$

$$AE = AB - BE = AB - BC; \tag{243}$$

and

$$\left.\begin{array}{l}(\text{fig. } 10.)\ AC' = AP - PC' = AP - PC, \\ (\text{fig. } 11.)\ AC' = AP + PC' = AP + PC;\end{array}\right\} \tag{244}$$

which, being substituted in (241), give

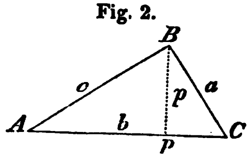
$$(\text{fig. } 10.)\ AC : AB + BC : : AB - BC : AP - PC, \tag{245}$$

$$(\text{fig. } 11.)\ AC : AB + BC : : AB - BC : AP + PC; \tag{246}$$

the proportions to be demonstrated (240).

60. *Problem.* To solve a triangle when its three sides are given.

Fig. 2.

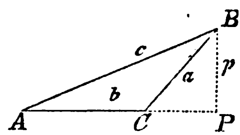


Solution. On the side b (figs. 2 and 3.) let fall the perpendicular BP.

Then, by (245) and (246),

$$\left.\begin{array}{l}(\text{fig. } 2.)\ b : c + a : : c - a : : PA - PC, \\ (\text{fig. } 3.)\ b : c + a : : c - a : : PA + PC.\end{array}\right\} \tag{247}$$

Fig. 3.



These proportions give the difference of the segments (fig. 2.), or their sum (fig. 3.). Then, adding the half difference to the half sum, that is, half the first to half the fourth term of (247), we obtain the larger segment corresponding to the larger of the two sides a and c. And, subtracting the half difference from the half sum, that is, taking half the difference be- (248)

tween the first and fourth terms of (247), we obtain the smaller segment.

Then, in triangles BCP and ABP, by (5) and (195), we have

(249) $$\cos. A = \frac{AP}{c};$$

and

(250) $$\begin{cases} \text{(fig. 2.)} \quad \cos. C = \dfrac{PC}{a}, \\ \text{(fig. 3.)} \quad \cos. C = -\cos. BCP = -\dfrac{PC}{a}. \end{cases}$$

(251) The third angle B is found by subtracting the sum of A and C from 180°.

61. *Corollary.* From (247), we have

(252) $$\begin{cases} \text{(fig. 2.)} \; PA - PC = \dfrac{(c+a)\,(c-a)}{b} = \dfrac{c^2 - a^2}{b}, \\ \text{(fig. 3.)} \; PA + PC = \dfrac{(c+a)\,(c-a)}{b} = \dfrac{c^2 - a^2}{b}; \end{cases}$$

which, added to

(253) $$\begin{cases} \text{(fig. 2.)} \; PA + PC = AC = b, \\ \text{(fig. 3.)} \; PA - PC = AC = b, \end{cases}$$

gives

(254) $$2\,PA = \frac{c^2 - a^2}{b} + b = \frac{b^2 + c^2 - a^2}{b}.$$

Hence

(255) $$PA = \frac{b^2 + c^2 - a^2}{2\,b},$$

and

(256) $$\cos. A = \frac{PA}{c} = \frac{b^2 + c^2 - a^2}{2\,b\,c}.$$

62. *Corollary.* Add unity to both sides of (256), and we have

$$1 + \cos. A = \frac{b^2 + c^2 - a^2}{2bc} + 1 = \frac{b^2 + 2bc + c^2 - a^2}{2bc}, \quad (257)$$

$$= \frac{(b + c)^2 - a^2}{2bc}. \quad (258)$$

Since the numerator of (258) is the difference of two squares, it may be separated into two factors, and we have

$$1 + \cos. A = \frac{(b + c + a)(b + c - a)}{2bc}. \quad (259)$$

Now, representing half the sum of the three sides of a triangle by s, we have

$$2s = a + b + c, \quad (260)$$

and

$$2s - 2a = 2(s - a) = a + b + c - 2a = b + c - a. \quad (261)$$

If we substitute these values in the numerator of the second member of (259), it becomes

$$1 + \cos. A = \frac{4s(s - a)}{2bc} = \frac{2s(s - a)}{bc}. \quad (262)$$

But, putting $C = A$ in (140),

$$1 + \cos. A = 2(\cos. \tfrac{1}{2} A)^2. \quad (263)$$

Hence

$$2(\cos. \tfrac{1}{2} A)^2 = \frac{2s(s - a)}{bc}, \quad (264)$$

or

$$(\cos. \tfrac{1}{2} A)^2 = \frac{s(s - a)}{bc}; \quad (265)$$

$$\cos. \tfrac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}}, \quad (266)$$

by logarithms,

(267) $$\left\{ \log.\cos.\tfrac{1}{2}A = \tfrac{1}{2}\begin{bmatrix}\log.s+\log.(s-a)+(\text{ar.co.})\\ \log.b+(\text{ar.co.})\log.c.\end{bmatrix}\right.$$

In the same way, we have

(268) $$\cos.\tfrac{1}{2}B = \sqrt{\frac{s\,(s-b)}{a\,c}};$$

(269) $$\cos.\tfrac{1}{2}C = \sqrt{\frac{s\,(s-c)}{a\,b}}.$$

63. *Corollary.* Subtract both sides of (256) from unity, and we have

(270) $$1-\cos.A = 1-\frac{b^2+c^2-a^2}{2\,b\,c} = \frac{a^2+2\,b\,c-b^2-c^2}{2\,b\,c},$$

(271) $$= \frac{a^2-(b-c)^2}{2\,b\,c}.$$

Since the numerator of (271) is the difference of two squares, it may be separated into two factors, as follows,

(272) $$1-\cos.A = \frac{(a-b+c)\,(a+b-c)}{2\,b\,c}.$$

But from (260)

(273) $$2\,s-2\,b = 2\,(s-b) = a+b+c-2\,b = a-b+c,$$

(274) $$2\,s-2\,c = 2\,(s-c) = a+b+c-2\,c = a+b-c.$$

If we substitute these values in the numerator of (272), it becomes

(275) $$1-\cos.A = \frac{4\,(s-b)\,(s-c)}{2\,b\,c} = \frac{2\,(s-b)\,(s-c)}{b\,c},$$

which, multiplied by (262), becomes

(276) $$1-(\cos.A)^2 = \frac{4\,s\,(s-a)\,(s-b\,(s-c)}{b^2\,c^2}.$$

But from (13)

$$1 - (\cos. A)^2 = (\sin. A)^2. \qquad (277)$$

Hence

$$(\sin. A)^2 = \frac{4\, s\, (s-a)\, (s-b)\, (s-c)}{b^2\, c^2}. \qquad (278)$$

or

$$\sin. A = \frac{2 \sqrt{s\, (s-a)\, (s-b)\, (s-c)}}{b\, c}. \qquad (279)$$

EXAMPLES.

1. Given the three sides of a triangle equal to 12.348, 13.561, 14.091; to solve the triangle.

Solution. First Method.

Make (fig. 2.) $a = 12.348$ $b = 13.561$
$c = 14.091.$

Then by (247), (249) and (250),

b	$= 13.561$ (ar. co.)	8.86771	
$c + a$	$= 26.439$	1.42224	
$c - a$	$= 1.743$	0.24130	
$PA - PC$	$= 3.398$	0.53125	
PA	$= 8.479$	0.92834	
PC	$= 5.081$		0.70595
c	$= 14.091$ (ar.co.)	8.85106	
a	$= 12.348$		(ar.co.) 8.90840
A	$= 53° 0'$ cos.	9.77940,	
C	$= 65° 42'$		cos. 9.61435.

$B = 180° - (A + C) = 180° - 118° 42' = 61° 18'.$

Second Method.

By (266), (268) and (269),

$a = 12.348$		(ar.co.) 8.90840	(ar.co.) 8.90840
$b = 13.561$	(ar. co.) 8.86771		(ar.co.) 8.86771
$c = 14.091$	(ar. co.) 8.85106	(ar.co.) 8.85106	
$s = 20.000$	1.30103	1.30103	1.30103
$s-a = 7.652$	0.88377		
$s-b = 6.439$		0.80882	
$s-c = 5.909$			0.77151
	2 | 19.90357	2 | 19.86931	2 | 19.84865
cos.	9.95179	9.93466	9.92433

$$\tfrac{1}{2} A = 26^\circ\, 30', \ \tfrac{1}{2} B = 30^\circ\, 39', \ \tfrac{1}{2} C = 32^\circ\, 51';$$

$$A = 53^\circ\ \ 0', \quad B = 61^\circ\, 18', \quad C = 65^\circ\, 42'.$$

Ans. The angles $= \begin{cases} 53^\circ\ \ 0', \\ 61^\circ\ 18', \\ 65^\circ\ 42'. \end{cases}$

2. Given the three sides of a triangle equal to 17.856, 13.349 and 11.111; to solve the triangle.

Ans. The angles $= \begin{cases} 93^\circ\ 20', \\ 48^\circ\ 16', \\ 38^\circ\ 24'. \end{cases}$

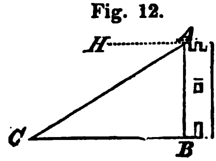
CHAPTER VI.

Application of Plane Trigonometry to the measurement of Heights and Distances.

64. ***Problem.*** To determine the height of a vertical tower, situated on a horizontal plane.

Solution. Suppose AB (fig. 12.) is the tower, whose height is to be determined. Measure off the distance BC on the horizontal plane of any convenient length. At the point C observe the angle of elevation ACB.

Fig. 12.



We have, then, given in the right triangle ACB the angle C and the base BC, as in problem, article 28, and the leg AB is found by (62),

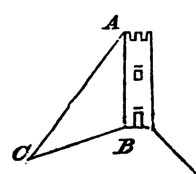
$$AB = BC \times \text{tang. } C. \qquad (280)$$

EXAMPLE. At the distance $BC = 95$ feet (fig. 12.) from the tower AB, the angle of elevation of the tower is found to be $C = 48° \; 19'$. Required the height of the tower. *Ans.* $AB = 106.69$ feet.

65. ***Problem.*** To find the height of a vertical tower situated on an inclined plane.

Solution. Let AB (fig. 13.) be the tower situated on the inclined plane BC. Observe the angle B, which the tower makes with the plane. Measure off the distance BC of any convenient length. Observe the angle C, made by a line drawn to the top of the tower with BC.

Fig. 13.



In the oblique triangle ABC, there are given the side BC and the two adjacent angles B and C as in problem, article 55.

By (212), we have

(281) $$A = 180^\circ - (B + C),$$

and by (214),

(282) $$\sin. A : \sin. C :: BC : AB;$$

whence

(283) $$AB = \frac{BC \times \sin. C}{\sin. A}.$$

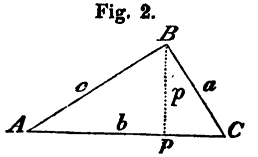
Example. Given (fig. 12.) $BC = 89$ feet, $B = 113^\circ\ 12'$, $C = 23^\circ\ 27'$, to find AB.

Ans. $AB = 51.595$ feet.

66. *Problem.* To find the distance of an inaccessible object.

Solution. Let B (fig. 2.) be the point, the distance of which is to be determined, and A the place of the observer. Measure off the distance AC of any convenient length and observe the angles A and C.

Fig. 2.



Then

(284) $$B = 180^\circ - (A + C),$$

and by (203),

(285) $$\sin. B : \sin. C :: AC : AB;$$

whence

(286) $$AB = \frac{AC \times \sin. C}{\sin. B}.$$

67. *Corollary.* The perpendicular distance BP of the point B from the line AC is found, in triangle ABP, by (50),

(287) $$BP = AB \times \sin. A.$$

EXAMPLE. Suppose two observers stationed at A and C (fig. 2.), on opposite sides of the cloud B, to observe the angles of elevation $A = 44° 56'$ and $C = 36° 4'$, their distance apart being $AC = 700$ feet. To find the distance of the cloud from the observer at A and its perpendicular altitude.

Ans. $AB = 417.2$ feet,
$BP = 294.7$ feet.

68. *Problem.* To find the distance of an object from the foot of a tower of known height, the observer being at the top of the tower.

Solution. Let the tower be AB (fig 12.) and the object C. Measure the angle of depression HAC.

Then, since

$$ACB = HAC,$$

we know in the triangle ACB the leg AB and the opposite angle C, as in problem, article 27. We have by (59),

$$BC = AB \times \text{cotan. } C. \quad (288)$$

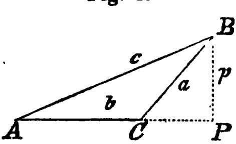
EXAMPLE. Given the height of the tower (fig. 12.) $AB = 150$ feet and the angle of depression $HAC = 17° 25'$, to find the distance BC.

Ans. $BC = 478.16$ feet.

69. *Problem.* To find the height of an inaccessible object above a horizontal plane and its distance from the observer.

Solution. Let B (fig. 3.) be the object, and C the place of the observer. Observe the angle of elevation BCP. Measure off, in a direct line from the object, the distance AC of any convenient length. At A observe the angle of elevation BAP.

Fig. 3.



As the exterior angle BCP of the triangle ABC, is the sum of the two opposite interior angles A and ABC, ABC must be the difference between BCP and A, that is,

(289) $$ABC = BCP - A.$$

Then, to find BC, in triangle ABC, by (203)

(290) $$\sin. ABC : \sin. A :: AC : BC;$$

whence

(291) $$BC = \frac{AC \times \sin. A}{\sin. ABC}.$$

Lastly, to find BP and PC, we know, in the right triangle BCP, the hypothenuse BC and the angle BCP as in problem, article 26.

Hence by (50) and (52)

(292) $$BP = BC \times \sin. BCP;$$

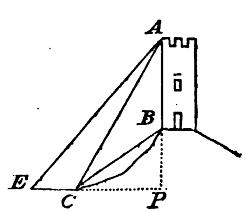
(293) $$PC = BC \times \cos. BCP.$$

Example. Given (fig. 3.) the angles of elevation $BCP = 68° 19'$ and $A = 32° 34'$, and the distance $AC = 546$ feet; to find BC, BP and PC.

Ans. $BC = 503.04$ feet,
$BP = 234.28$ feet,
$PC = 135.86$ feet.

70. ***Problem.*** To find the height of an inaccessible vertical tower situated on an elevation, the observer being on a horizontal plane.

Fig. 14.



Solution. Let AB (fig. 14.) be the tower and C the place of the observer. At C observe the angles of elevation of the top and of the bottom of the tower ACP and BCP. Measure in direct line from the tower any convenient distance

EC. At E observe the angle of elevation E of the top of the tower. Produce AB and EC to meet at P.

As ACP is an exterior angle of the triangle ACE, it is equal to the sum of E and CAE. Therefore CAE is the difference between ACP and E, or

$$CAE = ACP - E. \tag{294}$$

Then in triangle ACE, by (203),

$$\sin. CAE : \sin. E :: CE : AC, \tag{295}$$

whence

$$AC = \frac{CE \times \sin. E}{\sin. CAE}. \tag{296}$$

Again,

$$CBP = 90^\circ - BCP, \tag{297}$$

and

$$ABC = 180^\circ - CBP = 90^\circ + BCP. \tag{298}$$

Also

$$ACB = ACP - ACP. \tag{299}$$

Hence in triangle ABC, by (203),

$$\sin. ABC : \sin. ACB :: AC : AB, \tag{300}$$

and

$$AB = \frac{AC \times \sin. ACB}{\sin. ABC}. \tag{301}$$

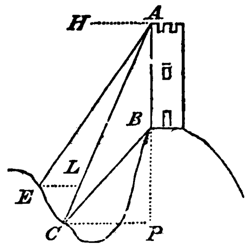
EXAMPLE. Given (fig. 13.) the angles $BCP = 44^\circ\ 32'$, $ACP = 56^\circ\ 29'$, and $E = 48^\circ\ 28'$, and the distance $CE = 300$ feet; to find AB.

Ans. $AB = 467.73$ feet.

71. *Problem.* To find the height of an inaccessible vertical tower situated on an elevation, the observer being on an inclined plane.

Solution. Let AB (fig. 15.) be the tower, and C the place of the observation on the inclined plane CE. At C observe the angles of elevation of the top and of the bottom of the tower BCP and ACP. Measure in a direct line from the tower the distance EC of any convenient length. At E observe the angle of elevation AEL of the top of the tower, and the angle of depression LEC of the former station C.

Fig. 15.



Then

(302) $$AEC = AEL + LEC.$$

Through A draw the horizontal line AH. Then the angles HAE and AEL are equal, being alternate internal angles, as are also HAC and ACP, and the angle EAC is the difference of HAE and HAC, or

(303) $$EAC = ACP - AEL.$$

Then, in the triangle CAE, from (203)

(304) $$\sin. EAC : \sin. AEC : : EC : AC,$$

whence

(305) $$AC = \frac{EC \times \sin. AEC}{\sin. EAC}.$$

Again

(306) $$CBP = 90^\circ - BCP,$$

and

(307) $$ABC = 180^\circ - CBP = 90^\circ + BCP;$$

also

(308) $$ACB = ACP - BCP.$$

Hence, in triangle ABC, from (203),

$$\sin. ABC : \sin. ACB :: AC : AB, \tag{309}$$

and

$$AB = \frac{AC \times \sin. ACB}{\sin. ABC}. \tag{310}$$

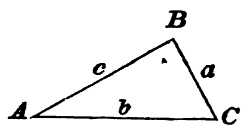
EXAMPLE. Given (fig. 14.) the angles $BCP = 45° 17'$, $ACP = 47° 27'$, $AEL = 46° 20'$, and $LEC = 10° 10'$, and the distance $EC = 400$ feet; to find the height of the tower.

Ans. $AB = 919.68$ feet.

72. *Problem.* To find the distance apart of two objects separated by an impassable barrier.

Solution. Let A and B (fig. 1.) be the objects; the distance of which from each other is sought. Measure the distances from any point C to both A and B, and also observe the angle C.

Fig. 1.



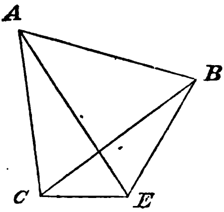
To find AB, we know in the triangle ABC the two sides AC and BC and the included angle C as in problem, article 58, and it may be solved by the method there explained as in (231), (233), (236), and (238).

EXAMPLE. Given (fig. 1.) $AC = 198$ feet, $BC = 200$ feet, and the angle $C = 60°$; to find AB.

Ans. $AB = 199$ feet.

73. *Problem.* To find the distance apart of two inaccessible objects situated in the same plane with the observer.

Fig. 16.



Solution. Let A and B (fig. 16.) be the two inaccessible objects, and C the place of the observer. Measure off any convenient distance CE. Observe the angles ACE, BCE, AEC and BEC.

In the triangle AEC, we have

(311) $$CAE = 180° - (AEC + ACE).$$

and by (203)

(312) $$\sin. CAE : \sin. AEC :: EC : AC,$$

whence

(313) $$AC = \frac{EC \times \sin. AEC}{\sin CAE}.$$

Again, in triangle BEC, we have

(314) $$EBC = 180° - (BEC + BCE),$$

and by (203)

(315) $$\sin. EBC : \sin BEC :: EC : BC,$$

whence

(316) $$BC = \frac{EC \times \sin. BEC}{\sin. EBC}.$$

Lastly, in triangle ABC, we know the two sides AC and BC and the included angle for

(317) $$ACB = ACE - BCE.$$

Hence by (231), (233), (236), and (203)

(318) $$CAB + CBA = 180° - ACB,$$

(319) $$AC + BC : AC - BC :: \text{tang.} \tfrac{1}{2} (CBA + CAB) : \text{tang.} \tfrac{1}{2} (CBA - CAB),$$

(320) $$CAB = \tfrac{1}{2} (CAB + CBA) - \tfrac{1}{2} (CBA - CAB),$$

(321) $$\sin. CAB : \sin. ACB :: BC : AB.$$

EXAMPLE. Given (fig. 15.) the angles $ACE = 27° \cdot 10'$, $BCE = 47\ 10'$, $BEC = 37°\ 10'$, $AEC = 57°\ 10'$, and the distance $EC = 500$ feet, to find AB.

Ans. $AB = 171.91$ feet.

CHAPTER VII.

Application of Plane Trigonometry to Navigation.

SECTION I.

Definitions.

74. The daily revolution of the earth is performed round a straight line, passing through its centre, which is called the *earth's axis.*

The extremities of this axis on the surface of the earth are the *terrestrial poles,* one being the *north pole,* and the other the *south pole.*

The section of the earth, made by a plane passing through its centre and perpendicular to its axis, is the *terrestrial equator.*

Parallels of latitude, are the circumferences of small circles, the planes of which are parallel to the equator.

Meridians are the semicircumferences of great circles, which pass from one pole to the other.

The *first meridian* is one arbitrarily assumed, to which all others are referred. In most countries, that has been taken as the first meridian which passes through the capital of the country. But, in the

United States, we have usually adhered to the English custom, and we consider the meridian, which passes through the Observatory of Greenwich, as the first meridian.

75. The *latitude* of a place is its angular distance from the equator, the vertex of the angle being at the centre of the earth; or, it is the arc of the meridian, passing through the place, which is comprehended between the place and the equator.

Latitude is reckoned north and south of the equator from 0° to 90°.

The *difference of latitude* of two places is the angular distance between the parallels of latitude in which they are respectively situated, the vertex of the angle being at the centre of the earth; or it is the arc of a meridian which is comprehended between the parallels of latitude.

The difference of latitude of two places is equal to the difference of their latitudes, if they are on the
(322) same side of the equator; and to the sum of their latitudes, if they are on opposite sides of the equator.

76. The *longitude* of a place is the angle made by the plane of the first meridian with the plane of the meridian passing through the place; or it is the arc of the equator comprehended between these two meridians.

Longitude is reckoned East and West of the first meridian from 0° to 180°.

The *difference of longitude* of two places is the angle contained between the planes of the meridians passing through the two places; or it is the arc of the equator comprehended between these two meridians.

The difference of longitude of two places is equal (323)
to the difference of their longitudes, if they are on the same side of the first meridian; and to the sum of their longitudes, if they are on opposite sides of the first meridian, unless their sum be greater than 180°; in which case the sum must be subtracted from 360° to give the difference of longitude.

77. The *distance* between two places in Navigation is the portion of a curve which would be described by a ship, sailing from one place to the other in a path, which crosses every meridian at the same angle.

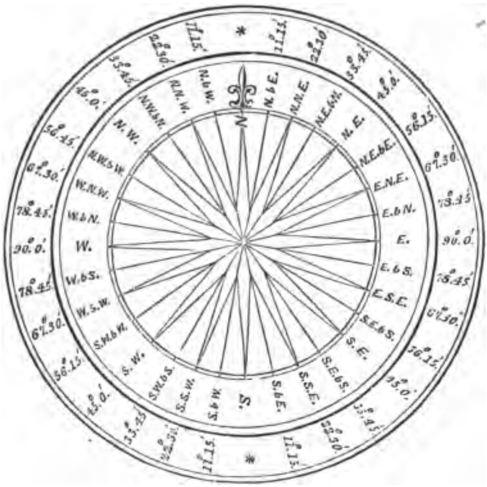
The *course* of the ship, or the *bearing* of the two places from each other, is the angle which the ship's path makes with the meridian.

The *departure* of two places is the distance of either from the meridian of the other, when they are (324)
so near each other that the earth's surface may be considered as plane and its curvature neglected. But, if the two places are at a great distance from each other, the distance is to be divided into small portions, (325)
and the *departure* of the two places is the sum of the departures corresponding to all these portions.

78. Instead of dividing the quadrant into 90 degrees, navigators are in the habit of dividing it into eight equal parts called *points*; and of subdividing the points into halves and quarters. A point, therefore, is equal to one eighth of 90° or to 11° 15′.

Names are given to the directions determined by the different points as in the diagram (fig. 17.), which represents the face of the card of the *Mariner's Compass.*

Fig. 17.



The *Mariner's Compass* consists of this card, attached to a magnetic needle, which has the property of constantly pointing toward the north and thereby determining the ship's course.

SECTION II.

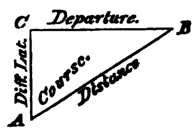
Plane Sailing.

79. The object of *Plane Sailing* is to calculate the Distance, Course or Bearing, Difference of Latitude and Departure, when either two of them are known.

80. *Problem.* To find the difference of latitude and departure, when the distance and course are known.

Solution. *First.* When the distance is so small that the curvature of the earth's surface may be neglected. Let AB (fig. 18.) be the distance. Draw through A the meridian AC, and let fall on it the perpendicular BC. The angle A is the course, AC is the difference of latitude, and BC is the departure.

Fig. 18.



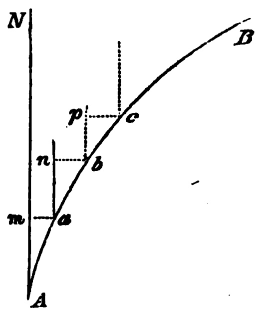
Then, by (50) and (53),

$$\text{Diff. of lat.} = \text{dist.} \times \cos. \text{course}; \quad (326)$$

$$\text{Departure} = \text{dist.} \times \sin. \text{course}. \quad (327)$$

Secondly. When the distance is great, as AB (fig. 19), then divide it into smaller portions, as $A\,a$, $a\,b$, $b\,c$, &c. Through the points of division, draw the meridians AN, $a\,n$, $b\,p$, &c. Let fall the perpendiculars $a\,m$, $b\,n$, $c\,p$, &c. Then, as the course is every where the same, each of the angles $m\,A\,a$, $n\,a\,b$, $p\,b\,c$, &c. is equal to the angle A or the course. Moreover, the distances $A\,m$, $a\,n$, $b\,p$, &c. are the differences of latitude respectively of A and a, a and b, b and c, &c. Also $a\,m$, $b\,n$, $c\,p$, &c. are the departures of the points A and a, a and b, b and c, &c. Therefore, as the difference of latitude of A and B is evidently equal to the sum of these partial differences of latitude;

Fig. 19.



and as the departure of A and B is by (325) equal to the sum of the partial departures, we have

(328) $$\text{Diff. of lat.} = Am + an + bp + \&c.$$

(329) $$\text{Departure} = am + bn + cp + \&c.$$

But the right triangles mAa, nab, pbc, &c. give by (326) and (327)

(330) $$\begin{cases} Am = Aa \times \cos.\ \text{course}, \\ am = Aa \times \sin.\ \text{course}; \end{cases}$$

(331) $$\begin{cases} an = ab \times \cos.\ \text{course}, \\ bn = ab \times \sin.\ \text{course}; \end{cases}$$

(332) $$\begin{cases} bp = bc \times \cos.\ \text{course}, \\ cp = bc \times \sin.\ \text{course}. \end{cases}$$

&c. &c.

The sum of the first of each pair of equations (330), (331), (332), &c. gives (333) by means of (328). And the sum of the second of each pair of the same equations gives (334) by means of (329)

(333) $$\begin{aligned} \text{Diff. of lat.} &= Am + an + bc + \&c. \\ &= (Aa + ab + bc + \&c.) \times \cos.\ \text{course}, \end{aligned}$$

(334) $$\begin{aligned} \text{Departure} &= am + bn + cp + \&c. \\ &= (Aa + ab + bc + \&c.) \times \sin.\ \text{course}. \end{aligned}$$

But

(335) $$Aa + ab + bc + \&c. = AB = \text{distance}.$$

Hence

(336) $$\text{Diff. of lat.} = \text{dist.} \times \cos.\ \text{course},$$

(337) $$\text{Departure} = \text{dist.} \times \sin.\ \text{course};$$

precisely the same with (326) and (327). This shows that the method of calculating the difference of lati-

(338) tude and departure is the same for all distances, and

that all the problems of Plane Sailing may be solved by the right triangle (fig. 18.).

Example. Suppose a ship to sail a distance of 2345 miles from latitude 3° 45′ South, on a course N. by E. To find the latitude at which the ship arrives and the departure which she makes.

Ans. Latitude sought = 34° 35′ N.
Departure = 457.3 miles.

81. ***Problem.*** To find the distance and difference of latitude, when the course and departure are known.

Solution. There are given (fig. 18.) the angle A and the side BC. Hence, by (56) and (59),

$$\text{Distance} = \text{departure} \times \text{cosec. course}, \qquad (339)$$

$$\text{Diff. of lat.} = \text{departure} \times \text{cotan. course}. \qquad (340)$$

Example. Suppose a ship to sail from latitude 62° 19′ N., upon a course W. N. W., till it makes a departure of 1000 miles. To find the latitude at which the ship arrives and the distance which it sails.

Ans. Latitude sought = 69° 15′ N.
Distance = 1082.4 miles.

82. ***Problem.*** To find the distance and departure, when the course and difference of latitude are known.

Solution. There are given (fig. 18.) the angle A and the side AC. Then, by (61) and (62),

$$\text{Distance} = \text{diff. of lat.} \times \text{sec. course}, \qquad (341)$$

$$\text{Departure} = \text{diff. of lat.} \times \text{tang. course}. \qquad (342)$$

Example. Given the latitudes of two places, of the one 67° 45′ N., of the other 7° 27′ S., and the

bearing of the second from the first S. S. E., to find their distance apart and their departure.

Ans. Distance $= 4883.7$ miles,
Departure $= 1868.9$ miles.

83. ***Problem.*** To find the course and difference of latitude when the distance and departure are known.

Solution. There are given (fig. 18.) the hypothenuse AB and the side BC. Then, by (65) and (69),

(343) $$\text{sin. course} = \frac{\text{departure}}{\text{distance}}.$$

(344) $$\text{Diff. of lat.} = \sqrt{(\text{Dist.})^2 - (\text{departure})^2}.$$

Example. Suppose a vessel to sail from latitude 72° 3′ S., a distance of 2000 miles on a course between the north and the west, until she makes a departure of 1000 miles. To find her course and the latitude at which she arrives.

Ans. Course $=$ N. 30° W.
Latitude sought $=$ 43° 10′ S.

84. ***Problem.*** To find the course and departure when the distance and difference of latitude are known.

Solution. There are given (fig. 18.) the hypothenuse AB and the leg AC. Then, by (65) and (69),

(345) $$\text{cos. course} = \frac{\text{diff. of lat.}}{\text{distance.}},$$

(346) $$\text{Departure} = \sqrt{(\text{dist.})^2 - (\text{diff. of lat})^2}.$$

Example. Given the distance between two places 1500 miles, the latitude of one being 16° 25′ N., that of the other being 17° 25′ N., and the second being

to the east of the first; to find the bearing of the second from the first and their departure.

Ans. Bearing = N. 87° 42′ E.
Departure = 1498.8 miles.

85. *Problem.* To find the course and distance when the departure and difference of latitude are known.

Solution. There are given (fig. 18.) the legs AC and BC. Then, by (71) and (74),

$$\text{tang. course} = \frac{\text{departure}}{\text{diff. of lat.}}, \tag{347}$$

$$\text{Dist.} = \text{diff. of lat.} \times \text{sec. course.} \tag{348}$$

Example. Given the departure of two places 1750 miles, and their latitudes, that of one 4° 10′ N., that of the other 4° 10′ S., the second being to the west of the first; to find their distance apart and the bearing of the second from the first.

Ans. Distance = 1821.4 miles,
Bearing = S. 74° 3′ W.

SECTION III.

Traverse Sailing.

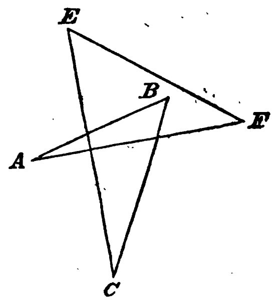
86. The object of Traverse Sailing is to solve the following problem by the principles of Plane Sailing. Traverse Sailing can only be used where the distances sailed are so small that the earth's surface may be considered as a plane.

87. *Problem.* To reduce several successive tracks of a ship to one; that is, to find the single track,

leading to the place, which the ship has actually reached, by sailing on several successive tracks.

Solution. Suppose the vessel to start from the point A (fig. 20.) and to sail, first from A to B, then from B to C, then from C to E, and lastly from E to F; to find the bearing and distance of F from A. Call the differences of latitude, corresponding to the 1st, 2d, 3d, and 4th tracks, the 1st, 2d, 3d, and 4th differences of latitude; and call the corresponding departures the 1st, 2d, 3d, and 4th departures. Then we need no demonstration to prove that,

Fig. 20.



(349) Diff. of lat. of A and F = 1st diff. of lat. — 2d diff. of lat. + 3d diff. of lat. — 4th diff., &c.;

or that the difference of latitude of A and F is found by *taking the sum of the differences of latitude corresponding to the northerly courses, and also the sum*
(350) *of those corresponding to the southerly courses, and the difference of these sums is the required difference of latitude.*

By neglecting the earth's curvature, we also have,

(351) Departure of A and F = 1st dep. — 2d dep. — 3d dep. + 4th dep.,

or the departure of A and F is found by *taking the*

sum of the departures corresponding to the easterly courses, also the sum of those corresponding to the westerly courses; and the difference of these sums is the required departure. (352)

Having thus found the difference of latitude and departure of A and F, their distance and bearing are found by (347) and (348),

$$\text{tang. bearing} = \frac{\text{departure}}{\text{diff. of lat.}}, \tag{353}$$

$$\text{Dist.} = \text{diff. of lat.} \times \text{sec. course.} \tag{354}$$

88. The calculations of traverse sailing are usually put into a tabular form, as in the following example. In the *first* column of the table are the numbers of the courses; in the *second* and *third* columns are the courses and distances; in the *fourth* and *fifth* columns are the differences of latitude, the column, headed N, corresponding to the northerly courses, and that headed S, to the southerly courses; in the *sixth* and *seventh* columns are the departures, the column, headed E, corresponding to the easterly courses, and that, headed W, to the westerly courses.

EXAMPLES.

1. Suppose a ship to sail on several successive tracks, in the order and with the courses and distances of the first three columns of the following table; find the bearing and distance of the place at which the ship arrives, from that from which she started.

Solution.

No.	Course.	Dist.	N.	S.	E.	W.
1	N. N. E.	30	27.7		11.5	
2	N. W.	80	56.6			56.6
3	West.	60				60.0
4	S. E. by S.	55		45.7	30.6	
5	North.	43	43.0			
6	S. by W.	152		149.1		29.7
	Sum of columns,		127.3	194.8	42.1	146.3
				127.3		42.1

Diff. of lat. = 67.5 S. dep. = 104.2W

Dep. = 104.2		2.01787		
Diff. of lat. = 67.5	(ar. co.)	8.17070		1.82930
Bearing = 57° 4′	tang.	0.18857	sec.	0.26467
Dist. = 124.1				2.09397

Ans. Bearing = S. 57° 4′ W.
Distance = 124.1 miles.

2. Suppose a ship to sail on the following successive tracks, South 10 miles, W. S. W. 25 miles, S. W. 30 miles, and West 20 miles.

Required the bearing and distance of the place at which the ship arrives, from that from which she departed.

Ans. Bearing = S. 57° 36′ W.
Distance = 76.2 miles.

SECTION IV.

Parallel Sailing.

89. *Parallel Sailing* considers only the case where the ship sails exactly east or west, and therefore remains constantly on the same parallel of latitude.

Its object is to find the change in longitude corresponding to the ship's track.

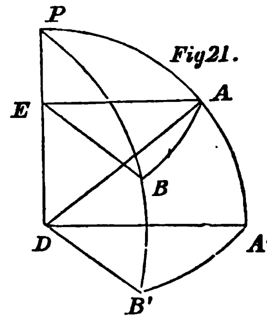


Fig 21.

90. *Problem.* Let AB (fig. 21.) be the distance sailed by the ship on the parallel of latitude AB. As the course is exactly east or west, the distance sailed must be itself equal to its departure. (355)

The latitude of the parallel is ADA' or AA'. The angle $AEB = A'DB'$, or the arc $A'B'$, is the difference of longitude. Denote the radius of the earth (356) $A'D = B'D = AD$ by R, and the radius of the parallel $AE = BE$ by r; also the circumference of the earth by C, and that of the parallel by c.

Since AB and $A'B'$ correspond to the equal angles AEB and $A'DB'$, they must be similar arcs, and give the proportion,

$$AB : A'B' : : c : C, \tag{357}$$

or

$$\text{Dep.} : \text{diff. of long.} : : c : C. \tag{358}$$

But, as circumferences are proportional to their radii,

$$c : C : : r : R. \tag{359}$$

Hence, leaving out the common ratio,

$$\text{Dep.} : \text{diff. of long.} : : r : R. \tag{360}$$

Putting the product of the extremes equal to that of the means,

(361) $$r \text{ diff. of long.} = R \text{ departure.}$$

But, in the triangle ADE, since

(362) $$DAE = ADA' = \text{latitude},$$

we have, from (53),

(363) $$r = R \times \text{cos. lat.},$$

which, substituted in (361), gives, if the result is divided by R,

(364) $$\text{Diff. of long.} \times \text{cos. lat.} = \text{departure.}$$

Hence, by (7),

(365) $$\text{Diff. of long.} = \frac{\text{departure}}{\text{cos. lat.}} = \text{dep.} \times \text{sec. lat.}$$

EXAMPLE. Suppose a vessel to sail from latitude 45° 54′ N., and longitude 56° 19′ W., a distance of 1000 miles exactly east. Required the longitude of the place at which it arrives.

Ans. Longitude sought = 32° 22′ W.

SECTION V.

Middle Latitude Sailing.

91. The object of *Middle Latitude Sailing* is to give an approximative method of calculating the difference of longitude, when the difference of latitude is small.

92. *Problem.* To find the difference of longitude by Middle Latitude Sailing, when the distance and course are known, and also the latitude of either extremity of the ship's track.

Solution. The difference of latitude and departure are found by (336) and (337),

$$\text{Diff. of lat.} = \text{dist.} \times \text{cos. course}; \qquad (366)$$
$$\text{Departure} = \text{dist.} \times \text{sin. course}. \qquad (367)$$

The difference of longitude may then be found by means of (365). But there is a difficulty with regard to the latitude to be used in (365); for, of the two extremities of the ship's track, the latitude of one is smaller, while the latitude of the other extremity is larger than the latitude of the rest of the track. Navigators have evaded this difficulty by using the *Middle Latitude* between the two, as sufficiently accurate when the difference of latitude is small. As the middle latitude is the arithmetical mean between the latitudes of the extremities, we have,

Middle lat. $= \frac{1}{2}$ sum of the lats. of the extremities of the track; (368)

and, by (365),

$$\text{Diff. of long.} = \frac{\text{departure}}{\text{cos. mid. lat.}} = \text{dep.} \times \text{sec. mid. lat.}, \qquad (369)$$

or, by substituting (367),

$$\text{Diff. of long.} = \text{dist.} \times \text{sin. course} \times \text{sec. mid. lat.} \qquad (370)$$

EXAMPLES.

1. Suppose a ship to sail from latitude 10° 29′ N., and longitude 30° 12′ E., a distance of 2000 miles, upon a course E. by S. Required the latitude and longitude of the place at which the ship arrives.

Solution.

Dist. = 2000.	3.30103		3.30103
Course = 7 points. cos.	9.29024	sin.	9.99157
Diff. of lat. = 390′	———		
= 6° 30′ S.	2.59127		
Given lat. = 10° 29′ N.	Mid. lat. = 7° 14′	sec.	0.00347
Req. lat. = 3° 59′ N.	Diff. of long. = 1977′		———
	= 32° 57′ E.		3.29607
Mid. lat. = 7° 14.′	Giv. long. = 30° 12′ E.		
	Required long. = 63° 9′ E.		

Ans. Required latitude = 3° 59′ N.;
Required longitude = 63° 9′ E.

2. Suppose a ship to sail from latitude 35° 25′ S., and longitude 14° 44′ W., a distance of 3500 miles, upon a course E. by S. Required the latitude and longitude of the place at which the ship arrives.

Ans. Required latitude = 46° 48′ S.
Required longitude = 61° 13′ E.

93. *Problem.* To find the distance and bearing of two places from each other, when their latitudes and longitudes are known, the difference of latitude being small.

Solution. From (369) multiplied by cos. mid. lat., we have,

(371) $$\text{Departure} = \text{diff. of long.} \times \text{cos. mid. lat.};$$

and then, from (347) and (348),

(372) $$\text{tang. bearing} = \frac{\text{departure}}{\text{diff. of lat.}};$$

(373) $$\text{Dist.} = \text{diff. of lat.} \times \text{sec. bearing}.$$

EXAMPLE. Given the latitude of Boston 42° 21′ N., and its longitude 70° 58′ W.; also the latitude of Paris 48° 50′ N., and its longitude 2° 15′ E.; to find the bearing and distance of Boston from Paris.

Ans. Bearing = S. 82° 47′ W.
Distance = 3097 miles.

SECTION VI.

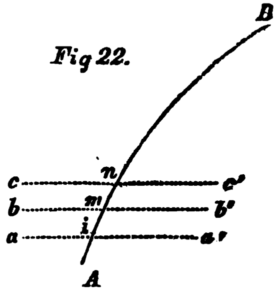
Mercator's Sailing.

94. The object of *Mercator's Sailing* is to give a method of calculating the difference of longitude, sufficiently accurate to be applied to any track whatever.

95. *Problem.* To find the difference of longitude by Mercator's Sailing, when the distance and course are known.

Solution. Let AB (fig. 22.) be the track of the ship. Draw parallels of latitude as $a\,a'$, $b\,b'$, $c\,c'$, &c., through every minute of latitude from A to B. These parallels divide the distance AB into the small portions $A\,i$, $i\,m$, $m\,n$, &c.

Fig 22.



(374)

Find the difference of longitude corresponding to each of these portions by means of (369), or by the following deduced from (369), by substituting (342),

(375) Diff. of long. = diff. of lat. × tang. course × sec. mid. lat.

But, in the present case, each difference of latitude is, by (374), just one minute; and the middle point of each portion differs so little in latitude from either of its extremities, that the latitude of either extremity may without practical error be substituted for the middle latitude; and (375) becomes,

(376) Diff. of long. = tang. course × sec. lat.

Thus, for each successive side, we have

(377)
1st diff. of long. = tang. course × sec. of lat. of parallel $a\,a'$,
2d diff. of long. = tang. course × sec. of lat. of parallel $b\,b'$,
3d diff. of long. = tang. course × sec. of lat. of parallel $c\,c'$,
&c. &c. &c.

The sum of these equations is,

(378) Diff. of long. of A and B = tang. course × sum of secants of all lats. from that of $a\,a'$ to that of B.

Suppose now the whole distance from the equator to the poles to be divided into intervals of latitude of one minute each. Suppose also a table to be constructed containing opposite each latitude the sum of
(379) the secants of all latitudes from the equator to the latitude itself. Such a table is called a table of *Meridional Parts*; and the number opposite each latitude is called the *Meridional Parts of that Latitude.*

So that,

Mer. parts of lat. = sum of secants of all lats. from 0° to the lat. (380)

Moreover the difference between the meridional parts of the latitudes of two places is called their *Meridional Difference of Latitude.*

Hence

Mer. diff. of lat. = diff. of mer. parts of lats., (381)

or, by (380),

Mer. diff. of lat. = sum of secants of all lats. from 0° to the greater lat. (382)
— sum of secants of all lats. from 0° to the less lat.
= sum of secants of all lats. above the less as far as the greater, (383)

or,

Mer. diff. of lat. of A and B = sum of secants of all lats. from that of a to that of B, (384)

which, substituted in (378), gives

Diff. of long. = tang. course × mer. diff. of lat. (385)

EXAMPLES.

1. Suppose a ship to sail from latitude 25° 4′ S., and longitude 105° 45′ W., a distance of 5000 miles upon a course N. W. Required the latitude and longitude of the place at which the ship arrives.

Solution.

Dist. $= 5000$			3.69897
Course $=$ 4 points		cos.	9.84949
Diff. of lat. $= 3536' = 58^\circ\ 56'$ N.			3.54846

Given lat. $= 25^\circ\ \ 4'$ S.

Required lat. $= 33^\circ\ 52'$ N.

Mer. parts of lat. $25^\circ\ \ 4'$ S. $= 1554$
Mer. parts of lat. $33^\circ\ 52'$ N. $= 2162$

Mer. diff. of lat. $= 3716$

Mer. diff. of lat. $= 3716$			3.57008
Course $=$ 4 points		tang.	0.00000
Diff. of long. $= 3716' = 61^\circ\ 56'$ W.			3.57008
Given long. $= 105^\circ\ 45'$ W.			

Required long. $= 167^\circ\ 41'$ W.

Ans. Required latitude $= 33^\circ\ 52'$ N.
Required longitude $= 167^\circ\ 41'$ W.

2. Suppose a ship to sail from latitude $1^\circ\ 5'$ N., and longitude 175° E., a distance of 5000 miles, on a course N. E. by E. Required the latitude and longitude of the place at which she arrives.

Ans. Required longitude $= 104^\circ\ 53'$ W.
Required latitude $= 47^\circ\ 23'$ N.

96. *Problem.* To find the distance and bearing of two places from each other, when their latitudes and longitudes are known.

Solution. Find the meridional difference of latitude by (381),

$$\text{Mer. diff. of lat.} = \text{diff. of mer. parts of lats.}; \tag{386}$$

and, then, the bearing from (385), divided by mer. diff. of lat.

$$\text{tang. bearing} = \frac{\text{diff. of long.}}{\text{mer. diff. of lat.}}; \tag{387}$$

and the distance from (341),

$$\text{dist.} = \text{diff. of lat.} \times \text{sec. bearing.} \tag{388}$$

Example. Given the latitude of St. Petersburgh 59° 56′ N., and its longitude 30° 25′ E.; also the latitude of Cape Horn 55° 58′ S., and longitude 67° 20′ W.; to find the bearing and distance of St. Petersburgh from Cape Horn.

Ans. Bearing = N. 34° 20′ E.
Distance = 8421 miles.

CHAPTER VIII.

Application of Plane Trigonometry to Surveying.

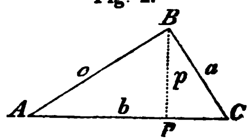
97. The object of *Surveying* is to determine the dimensions and areas of portions of the earth's surface. In the application of Plane Trigonometry, the portions of the earth are supposed to be so small that the curvature of the earth is neglected. They

are, in this case, nothing more than common fields bounded by lines either straight or curved.

98. *Problem.* To find the area of a triangular field, when its angles and one of its sides are known.

Solution. Let ABC (fig. 2.) be the triangle to be measured, and c the given side. The area of the triangle is equal to half the product of its base by its altitude, or

Fig. 2.



(389) $$\text{Area of } ABC = \tfrac{1}{2}\, bp.$$

But, by (203),

(390) $$\sin.\, b : \sin.\, B :: c : b,$$

whence

(391) $$b = \frac{c \sin.\, B}{\sin.\, C};$$

and, by (206),

(392) $$p = c \sin.\, A.$$

Substituting (391) and (392) in (389), we have

(393) $$\text{area of } ABC = \frac{c^2 \sin.\, A \sin.\, B}{2 \sin.\, C}.$$

Example. Given one side of a triangular field equal to 17.95 ch., and the adjacent angles equal to 100° and 70°; to find its area.

Ans. Required area = 85 A. 3 R. 16 r.

99. *Problem.* To find the area of a triangular field, when two of its sides and the included angle are known.

Solution. Let ABC (fig. 2.) be the triangle to be measured, b and c the given sides, and A the given angle. Then, by (389),

$$\text{area of } ABC = \tfrac{1}{2}\, b\, p, \tag{394}$$

and, by (392),

$$p = c \sin A. \tag{395}$$

Hence

$$\text{area of } ABC = \tfrac{1}{2}\, b\, c \sin A. \tag{396}$$

or, *the area of a triangle is equal to half the continued product of two of its sides and the sine of the included angle.* (397)

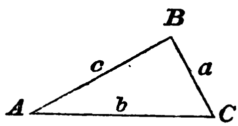
EXAMPLE. Given two sides of a triangular field equal to 12.34 ch. and 17.97 ch., and the included angle equal to 44° 56′; to find the area of the triangle.

Ans. Required area = 7 A. 3 R. 13 r.

100. *Problem.* To find the area of a triangular field, when its three sides are known.

Solution. Let ABC (fig. 1.) be the given triangle. Then, by (396),

Fig. 1.



$$\text{Area of } ABC = \tfrac{1}{2}\, b\, c \sin. A; \tag{398}$$

but, by (279),

$$\sin. A = \frac{2\sqrt{s.\,(s-a)\,(s-b)\,(s-c)}}{b\,c}, \tag{399}$$

in which s denotes the half sum of the three sides of the triangle.

Hence

(400) $$b\,c \sin. A = 2\sqrt{s\,(s-a)\,(s-b)\,(s-c)};$$

and, by (398),

(401) $$\text{area of } ABC = \sqrt{s\,(s-a)\,(s-b)\,(s-c)},$$

or, *to find the area of a triangular field, subtract*
(402) *each side separately from the half sum of the sides; and the square root of the continued product of the half sum and the three remainders is the required area.*

EXAMPLES.

1. Given the three sides of a triangular field, equal to 45.56 ch., 52.98 ch., and 61.22 ch.; to find its area.

Solution. In (fig. 1.), let $a = 45.56$ ch., $b = 52.98$ ch., $c = 61.22$ ch.

$2\,s =$	159.76 ch.	
$s =$	79.88 ch.	1.90244
$s - a =$	34.32 ch.	1.53555
$s - b =$	26.90 ch.	1.42975
$s - c =$	18.66 ch.	1.27091
		2 | 6.13865
Area of $ABC =$	1173. 1 sq. ch.	3.06932

Ans. Required area = 117 A. 1 R. 9 r.

2. Given the three sides of a triangular field equal

to 32.56 ch., 57.84 ch., and 44.44 ch.; to find its area.

Ans. 71 A. 3 R. 29 r.

101. *Problem.* To find the area of an irregular field bounded by straight lines.

First Method of Solution. Divide the field into triangles in any manner best suited to the nature of the ground. Measure all those sides and angles (403) which can be measured conveniently, remembering that three parts of each triangle, one of which is a side, must be known to determine it.

But it is desirable to measure more than three parts of each triangle, when it can be done; because the comparison of them with each other will often serve (404) to correct the errors of observation. Thus, if the three angles were measured, and their sum found to differ from 180°, it would show there was an error; and the error, if small, might be divided between the angles; but if the error was large, it would show the observations were inaccurate, and must be taken again.

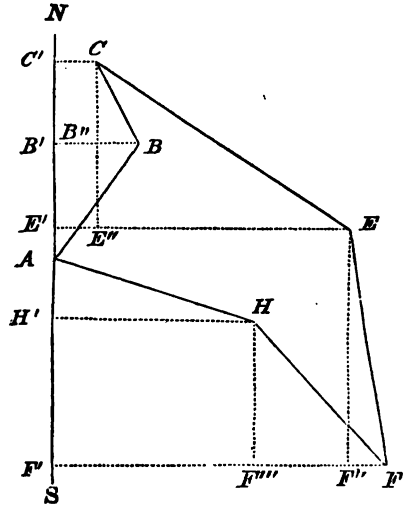
The area of each triangle is to be calculated by one of the preceding formulas, and the sum of the areas (405) of the triangles is the area of the whole field.

This method of solution is general, and may be applied to surfaces of any extent, provided each triangle is so small as not to be affected by the earth's curvature.

Second Method of Solution. Let *ABCEFH* (fig. 23.) be the field to be measured. Starting

from its most easterly or its most westerly point, the point A for instance, measure successively round the field the bearings and lengths of all its sides.

Fig. 23.



Through A draw the meridian NS, on which let fall the perpendicular BB', CC', EE', FF', and HH'. Also draw $CB''E''$, EF'', and HF''' parallel to NS.

Then the area of the required field is

(406) $$ABCEFH = AC'CEFF' - [AC'CB + AHFF'].$$

But

(407) $$AC'CEFF' = C'CEE' + E'EFF';$$

and

(408) $$AC'CB + AHFF' = C'CBB' + B'BA + AHH' + H'HFF'.$$

Hence, by (406),

$$ABCEFH = [C'CEE' + E'EFF'] - [C'CBB' + B'BA + AHH' + H'HFF']. \quad (409)$$

or doubling and changing a very little the order of the terms,

$$2\,ABCEFH = [2\,C'CEE' + 2\,E'EFF'] - [2\,B'BA + 2\,CC'BB' + 2\,H'HFF' + 2\,AHH']. \quad (410)$$

Again,

$$\left.\begin{aligned} 2\ B'BA &= BB' \times AB', \\ 2\ CC'BB' &= (BB' + CC') \times B'C', \\ 2\ C'CEE' &= (EE' + CC') \times E'C', \\ 2\ E'EFF' &= (EE' + FF') \times E'F', \\ 2\ H'HFF' &= (HH' + FF') \times H'F', \\ 2\ AHH' &= HH' \times AH'. \end{aligned}\right\} \quad (411)$$

So that the determination of the required area is now reduced to the calculation of the several lines in the second members of (411). But the rest of the solution may be more easily comprehended by means of the following table, which is precisely similar in its arrangement to the table actually used by surveyors, when calculating areas by this process.

Sides.	N.	S.	E.	W.	Dep.	Sum.	N. Areas.	S. Areas.
AB	AB′		BB′		BB′	BB′	BB′A	
BC	B′C′			BB″	CC′	BB′ + CC′	CC′BB′	
CE		C E′	EE″		EE′	CC′ + EE′		C′CEE′
EF		E′F′	FF″		FF′	EE′ + FF′		E′EFF′
FH	F′H′			FF‴	HH′	FF′ + HH′	H′HFF′	
HA	H′A			HH′	O	HH′	AHH′	

(412)

In the *first* column of the table are the successive sides of the field.

In the *second* and *third* columns are the differences of latitude of the several sides, the column headed N, corresponding to the sides running in a northerly direction, and that headed S, corresponding to those running in a southerly direction. These two columns are calculated by (326).

(413) Diff. of lat. = dist. $\times$ cos. bearing.

In the *fourth* and *fifth* columns are the departures of the several sides; the column headed E, corresponding to the sides running in an easterly direction, and that headed W, to those running in a westerly direction. These two columns are calculated by (327),

(414) Departure = dist. $\times$ sin. bearing.

In the *sixth* column, headed *departure*, are the departures of the several vertices which end each side of the field from the vertex *A*. This column is calculated from the two columns E, and W, in the following manner. *The first number in column departure is the same as the first in the two columns E,*
(415) *and W; and every other number in column departure is obtained by adding the corresponding number in columns E and W, if it is of the same column with the first number in those two columns, to the previous number in column departure; and by subtracting it, if it is of a different column.*

Thus,

$$\left.\begin{aligned}
BB' &= BB', \\
CC' &= B'B'' = BB' - BB'', \\
EE' &= E'E'' + EE'' = CC' + EE'', \\
FF' &= F'F'' + FF'' = EE' + FF'', \\
HH' &= F'F''' \qquad = FF' - FF''', \\
O &= HH' - HH'.
\end{aligned}\right\} \tag{416}$$

In the *seventh* column, headed *Sum*, are the first factors of the second members of (411). This column is calculated from column Departure in the following manner. *The first number in column Sum is the same as the first in column Departure; every other number* (417) *in column Sum is the sum of the corresponding number in column Departure added to the previous number in column Departure*, as is evident from simple inspection.

In the *eighth* and *ninth* columns are the values of the areas which compose the first members of (411). *These columns are calculated by multiplying the* (418) *members in column Sum by the corresponding numbers in columns N and S*, which contain the second factors of the second members of (411). *The products are written in the column of North Areas when the second factors are taken from column N, and in that of South Areas when the second factors are taken from column S.*

If we compare the columns of North and South Areas with (410), we find that all those areas which are preceded by the negative sign are the same with those in the column of North Areas; while all those, which are connected with the positive sign, belong to the column of South Areas. *To obtain therefore*

(419) *the value* of the second member of (410), that is, *of double the required area, we have only to find the difference between the sums of the columns of North and South Areas.*

102. *Corollary.* The columns N, S, E, and W, are those which would be calculated in Traverse Sailing, if a ship was supposed to start from the point *A* and proceed round the sides of the field till it returned to the point *A*. The difference of the sums of columns N and S is, then, by (350), the difference of latitude between the point from which the ship starts, and the point at which it arrives; and the difference of columns E and W is, by (352), the departure of the same two points. But as both the points are here the same, their difference of latitude and their departure must be nothing, or

(420) Sum of column N = sum of column S;

(421) Sum of column E = sum of column W.

But when, as is almost always the case, the sums of these columns differ from each other, the difference must arise from errors of observation. If the error is great, new observations must be taken; but, if it is small, it may be divided among the sides by the following proportion.

(422) *The sum of the sides* : *each side* : : *whole error* : *error corresponding to each side.*

The errors corresponding to the sides are then to be subtracted from the differences of latitude, or the de-
(423) partures which are in the larger column, and added to those which are in the smaller column.

EXAMPLES.

1. Given the bearings and lengths of the sides of a field, as in the three first columns of the following table; to find its area.

Solution. The first set of columns headed N, S, E, and W, of the table, is obtained from (413) and (414). The error of difference of latitude 11 ch. and that of departure 79 ch. are then apportioned among the sides by means of (422), and arranged in the columns headed *Cor. S.* and *Cor. E.* The second set of columns headed N, S, E, and W, is a corrected set obtained from (423). The columns headed *Dep.* and *Sum* are obtained from (415) and (417). The columns of *North* and *South Areas* are obtained from (418), and the area of the field is finally deduced from (419).

No.	Bearing.	Dist.	N.	S.	E.	W.	Cor. S.	Cor. E.	N.	S.	E.	W.	Dep.	Sum.	N. Areas.	S. Areas.
1	N. 45° W.	21 ch.	14.85			14.85	.01	.05	14.86			14.90	14.90	14.90	211.4140	
2	N. 24° E.	32 ch.	29.24		13.01		.02	.08	29.26		12.93		1.97	16.87	493.6162	
3	S. 86° W.	56 ch.		3.76		53.87	.04	.14		3.72		54.01	55.98	57.95		215.5740
4	South.	10 ch.		10.00			.01	.03		9.99		.03	56.01	111.99		1118.7801
5	S. 70° W.	11 ch.		3.76		10.34	.01	.03		3.75		10.37	66.38	122.39		459.9625
6	S. 20° E.	99 ch.		93.03	33.86		.06	.26		92.97	33.60		32.78	99.16		9218.9052
7	East.	6 ch.			6.00		.00	.02			5.98		26.80	59.58		
8	N. 22° E.	72 ch.	66.27		26.98		.04	.18	66.31		26.80		0.00	26.80	1997.1080	
			110.36	110.55	79.85	79.06	.19	.79	110.43	110.43	79.31	79.31			2482.1382	11013.2218
				110.36	79.06											2482.1382
				.19 S.	.79 E.											2) 8531.0836
																10) 4265.5418
																426.55418
																4
																2.21672
																40
																8.66880

Ans. 426 A. 2 R. 8 r.

2. Given the lengths and bearings of the sides of a field, as in the following table; to find its area.

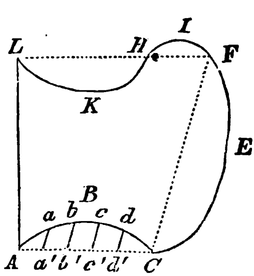
No.	Bearings.	Dist.
1	N. 17° E.	25 ch.
2	East.	28 ch.
3	South.	54 ch.
4	S. 4° W.	22 ch.
5	N. 33° W.	62 ch.

Ans. Required area = 173 A. 0 R. 36 r.

103. *Problem.* To find the area of a field bounded by sides, irregularly curved.

Solution. Let $ABCEF$ $HIKL$ (fig. 24.) be the field to be measured, the boundary $ABCEFHIKL$ being irregularly curved. Take any points C and F, so that by joining AC, CF and FL, the field $ACFL$, bounded by straight lines, may not differ much from the given field.

Fig. 24.



Find the area of $ACFL$, by either of the preceding methods, and then measure the parts included (224)
between the curved and the straight sides by the following method of *offsetts.*

Take the points a, b, c, d, so that the lines Aa, ab, bc, cd, dC may be sensibly straight. Let fall on AC the perpendiculars aa', bb' cc', dd'. Measure these

perpendiculars and also the distances Aa', $a'b'$, $b'c'$, $c'd'$, $d'C$.

The triangles Aaa', Cdd' and the trapeziums
(425) $aba'b'$, $bcb'c'$, $cdc'd'$ are then easily calculated, and
their sum is the area of ABC.

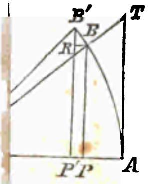
In the same way may the areas of CEF, FHI and IKL be calculated; and then the required area is found by the equation

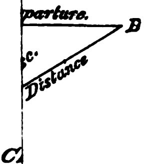
(426) $$ABCEFHIKL = ACFL - ABC + CEF + FHI - IKL.$$

EXAMPLE. Given (fig. 24.) $Aa' = 5$ ch., $a'b' = 2$ ch., $b'c' = 6$ ch., $c'd' = 1$ ch., $d'C = 4$ ch.; also $aa' = 3$ ch., $bb' = 2$ ch., $cc' = 2,5$ ch., $dd' = 1$ ch.; to find the area of ABC.

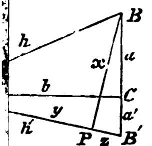
Ans. Required area $= 2$ A. 3 R. 36 r.

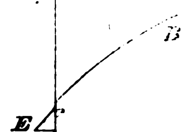
. 7.



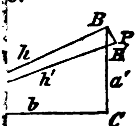


8.

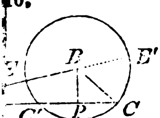




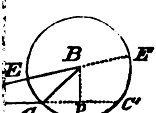
9.

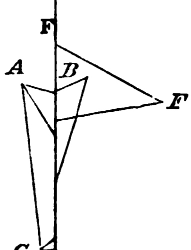


10.



1.





2.

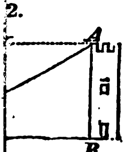


Fig. 21.

Fig. 22.

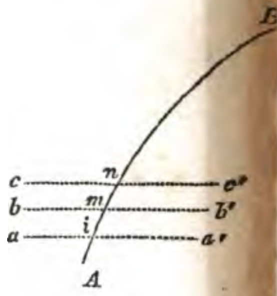


Fig. 24.